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FOUNDATIONS

★Bourbaki, N. Éléments de mathématique. V. Première partie: Les structures fondamentales de l'analyse. Livre III: Topologie générale. Chapitre V: Groupes à un paramètre. Chapitre VI: Espaces numériques et espaces projectifs. Chapitre VII: Les groupes additifs R^n. Chapitre VIII: Nombres complexes. Actualités Sci. Ind., no. 1029. Hermann et Cie., Paris, 1947. ii+132 pp.

Together with the preceding chapter [IV; Actualités Sci. Ind., no. 916, 1942; these Rev. 5, 102] this issue constitutes a fairly complete exposition of the so-called "foundations of analysis." Since, however, a good amount of algebra, general topology, topological groups, etc. is already available, the development is quite rapid and well integrated with earlier material. Chapter V is concerned with the additive group R of real numbers, its subgroups, quotient groups and homomorphisms. As an application the exponential function a^x $(1\neq a>0)$ is defined as an isomorphism f of R onto the multiplicative group of positive reals, with $f(1)=a$. The formal properties of the exponential and logarithmic functions are then obtained rapidly from the group properties. The section on the "measuring of magnitudes" which shows the proper place of the axiom of Archimedes is particularly beautiful.

In Chapter VII the additive groups R^n $(=R\times\cdots\times R$, n times) are studied. All closed subgroups of R^n and the respective quotient groups are enumerated. As applications, various approximation theorems of Kronecker are obtained. Chapters VI and VII are concerned with elementary properties of Euclidean spaces, projective spaces, complex numbers, complex projective spaces, quaternions, etc. These objects are treated as special cases of similar objects defined (in the book on algebra) over any field. To a large extent the contents consist in the introduction of a topology and a study of its elementary properties. The relevant chapters on algebra have not yet appeared. Also based on (yet unpublished) chapters on algebra is a discussion of angles and trigonometric functions. The issue contains a large number of excellent exercises. *S. Eilenberg* (New York, N. Y.).

Destouches-Février, Paulette. Esquisse d'une mathématique intuitioniste positive. C. R. Acad. Sci. Paris **225**, 1241–1243 (1947).

Griss [Nederl. Akad. Wetensch., Proc. 49, 1127–1133 = Indagationes Math. 8, 675–681 (1946); these Rev. 8, 307] has begun the construction of negationless intuitionistic mathematics. Starting from his theory of natural numbers, the author proposes to define negation by $\neg p =_d (p\rightarrow(1=2))$. Thus a logic will be obtained which will resemble the minimal calculus of Johansson [Compositio Math. 4, 119–136 (1936)]. The reviewer remarks that the sense as well as the formal properties of implication in Griss's system require further investigation, without which the value of this definition is questionable. *A. Heyting* (Amsterdam).

Quine, W. V. On universals. J. Symbolic Logic 12, 74–84 (1947).

It is well known that the usual interpretation of logical formulae, as appearing, e.g., in Principia Mathematica, depends on platonistic suppositions. The author rightly points out that the mere occurrence of predicate names, or even predicate variables, is not offensive to nominalism, as these may be meaningful in context, even if they are not considered as names of abstract entities or universals. Even abstraction in some cases turns out to be innocent. Generally speaking, however, quantification with regard to predicate or class variables is objectionable from a nominalistic point of view. This is demonstrated with respect to an axiomatization of the logic of classes, which the author borrows from A. Tarski. Consequently the author starts to restate this system in a form which is acceptable to nominalists.

Variable individuals are indicated by 'x,' 'y,' 'z,' $\cdots$; variable classes of individuals (classes of first type) by 'x^1,' 'y^1,' 'z^1,' $\cdots$; variable classes of classes of first type (classes of second type) by 'x^2,' 'y^2,' 'z^2,' $\cdots$; etc. Usual axioms of elementary logic are adopted, including quantification with regard to variable individuals. Moreover, an axiom and an axiom scheme for identity of individuals are needed: $x=x$; $x=y.\ Fx.\supset Fy$. Then the symbol 'ϵ' is introduced as follows:

$$x\epsilon\Lambda.=_{df}.\sim(x=x);\quad x\epsilon\{y\}.=_{df}.x=y;$$
$$x\epsilon\{y,z\}.=_{df}:x=y.\vee.x=z;\quad \text{etc.}$$

Limited quantification with regard to variable classes of first type is introduced by means of the following definition:

$$(x^1)_kFx^1=_{df}.F\Lambda.(y_1)(y_2)\cdots(y_k)F\{y_1,y_2,\cdots,y_k\},$$

where 'Fx^1' is any formula which contains 'x^1' only in positions preceded by 'ϵ.' Identity of classes of first type is also defined:

$$x^1=y^1.=_{df}.(z)(z\epsilon x^1.\equiv.z\epsilon y^1).$$

Now the author assumes that individuals (i.e., material objects, past, present, and future; point-events and spatiotemporally scattered totalities of point-events) are finite in number, hence not in excess of some finite number t. This is expressed as follows:

$$(\exists x_1)(\exists x_2)\cdots(\exists x_t)(y)(y=x_1.\vee.y=x_2.\vee.\cdots.\vee.y=x_t).$$

Then unlimited quantification with respect to classes of first type is defined by $(x^1)=_{df}(x^1)_t$. Now it is possible to give analogous definitions regarding appurtenance, limited quantification, identity and unlimited quantification with respect to classes of second type, etc. The validity of Tarski's axioms for the logic of classes may then be deduced.

The author recognizes that a logic which can be reconciled with nominalism only upon a highly speculative physical hypothesis of the type mentioned above is little better than a logic which cannot be reconciled with nominalism at all. As another objection he mentions the fact

that the symbolic formulation of this hypothesis would be too long to exist. *E. W. Beth* (Amsterdam).

Goodman, Nelson, and Quine, W. V. Steps toward a constructive nominalism. J. Symbolic Logic 12, 105–122 (1947).

After stating the purpose of nominalism [cf. the preceding review], the authors observe that in some cases statements referring to abstract entities are easily rephrased as statements about concrete objects. In other cases such rephrasing, though more difficult, turns out to be equally possible. The definition of ancestorhood in terms of parenthood according to Frege's method, for instance, depends on quantification with regard to a class variable, which is offensive to nominalists. In this case, however, the class-member relation may be replaced by the individual-member relation; the phrase 'b is an ancestor of c' is explained as follows:

$$b \neq c \,.\, (\exists u)\ \mathrm{Parent}\ bu \,.\, (\exists w)\ \mathrm{Parent}\ wc \,.\, (x)\{\mathrm{Part}\ cx \,.\ (y)(z)(\mathrm{Part}\ zx \,.\, \mathrm{Parent}\ yz \,.\supset \mathrm{Part}\ yx) \,.\supset \mathrm{Part}\ bx\}.$$

In order to explain the statement 'there are more cats than dogs,' the authors furthermore resort to the phrase 'x is a bit,' meaning: for every y, if y is a cat or a dog and is bigger than no other cat or dog, then neither is x bigger than y nor is y bigger than x. For brevity x is called a bit of z when x is a bit and is part of z. The above statement is then explained as meaning: Every individual that contains a bit of each cat is bigger than some individual that contains a bit of each dog.

These and similar devices are extensively used in building up a nominalistic syntax, which treats mathematical expressions as concrete objects: as actual strings of physical marks. The simple typographical shapes of the object language reduce to six: 'v,' '$'$,' '(,' '),' '$|$,' 'ϵ.' Accordingly, the authors introduce six shape predicates: 'Vee,' 'Ac,' 'LPar,' 'RPar,' 'Str,' 'Ep,' as well as the predicates 'C' ('Cxyz' means: x consists of y followed by z), 'Part' ('Partxy' means: x is contained entirely within y) and 'Bgr' ('Bgrxy' means: x is spatially bigger than y). The predicates of concatenation are introduced by means of the definitions:

$$\mathrm{C}xyzw = (\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzw); \quad \mathrm{C}xyzwu = (\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzwu);$$
$$\mathrm{C}xyzwus = (\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzwus).$$

As a matter of convenience, the authors introduce the notions of character:

$$\mathrm{Char}\ x = .\ \mathrm{Vee}\ x \vee \mathrm{Ac}\ x \vee \mathrm{LPar}\ x \vee \mathrm{RPar}\ x \vee \mathrm{Str}\ x \vee \mathrm{Ep}\ x,$$

and of inscription:

$$\mathrm{Insc}\ x = .\ \mathrm{Char}\ x \vee (\exists y)(\exists z)\mathrm{C}xyz.$$

Then the notions of initial segment, of final segment, of segment, are defined, as well as the notions of bit, of longer than, of equally long, and of likeness. These notions enable the authors to define the notions of a string of accents, of a variable, and of a string of quantifiers. They further call x a quantification of y if x consists of a string of quantifiers followed by y:

$$\mathrm{Qfn}\ xy = (\exists z)(\mathrm{Qfr\ String}\ z \,.\, \mathrm{C}xzy).$$

An atomic formula of the object language consists of two variables with an epsilon between them:

$$\mathrm{AtFmla}\ x = (\exists w)(\exists y)(\exists z)(\mathrm{Vbl}\ w \,.\, \mathrm{Ep}\ y \,.\, \mathrm{Vbl}\ z \,.\, \mathrm{C}xwyz).$$

The nonatomic formulas of the object language are constructed from the atomic formulas by quantification and alternative denial. Now it is possible to give a syntactical description of the axioms and of the rules of inference; the notion of substitution of course offers some difficulty. Finally the notions of proof and of theorem are defined.

In their conclusion, the authors observe that the gains which seem to have accrued to natural science from the use of mathematical formulas do not imply that those formulas are true statements. What is meaningful and true in platonistic mathematics is not the apparatus itself, but only its description: the rules by which it is constructed and run. Hence the importance of a syntax which is itself free from platonistic commitments.

The reviewer wishes to stress the basic similarity of these views to those held by L. Chwistek. *E. W. Beth.*

***Pólya, G. On patterns of plausible inference.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 277–288. Interscience Publishers, Inc., New York, 1948. $5.50.

In the first section of this paper the author discusses an example of inductive mathematical reasoning, dealing with the conjecture: Symmetrization decreases the capacity. In the second section he presents some patterns of plausible inference or "heuristic syllogisms," e.g.,

$$\begin{array}{l} \left\{\begin{array}{l} A\to B,\ A\to C,\ A\to D \\ B,\ C \text{ true} \\ D \text{ very different from } B,\ C \end{array}\right. \\ \underline{D \text{ true}} \\ A \text{ much more credible} \end{array}$$

and

$$\begin{array}{l} A \text{ analogous to } B \\ \underline{B \text{ true}} \\ A \text{ more credible.} \end{array}$$

The author observes that nonmathematicians, especially philosophers, have no idea of the possibility of such inductive mathematical reasoning. He argues that "inductive logic" should be merged into a larger domain which could be called the "logic of plausible inference."

E. W. Beth (Amsterdam).

Schrödinger, Erwin. The foundation of the theory of probability. II. Proc. Roy. Irish Acad. Sect. A. 51, 141–146 (1947).

[For part I see the same vol., 51–66 (1947); these Rev. 8, 559.] The author attempts to replace his multiplicative axiom (compound probability) by simpler and more self-evident assumptions. The method is based on comparisons of the general events whose probabilities are being considered with one another. No precise axioms of comparison are formulated, and the whole treatment is heuristic and suggestive rather than mathematically rigorous or logically complete.

An interesting suggestion is that the following general principle be adopted (it is made an essential part of the author's treatment): "If our knowledge suffices definitely to exclude from the numerical probability of a certain event all values save one, but definitely does not exclude that one, we admit it to have this value." *B. O. Koopman.*

Bouligand, Georges. Une épistémologie conforme à l'esprit de l'analyse classique. C. R. Acad. Sci. Paris 224, 1747–1749 (1947).

Rejecting the concept of mathematics as a perfect synthesis, the author introduces the notion S_t of the global

synthesis, established at a certain moment t; S_t is continually expanded and readjusted. A problem takes a mathematical character only at the moment when it is considered at the extraction, from certain categories, duly established within S_t, of an element answering certain given conditions. The practical mathematician will attempt to construct such an element. However, the adjustment of S_t contents itself with simple consistency; consequently, as soon as the excluded third is admitted as well as the product set of a set of mutually disjoinct sets, S_t is said to be consistent with the axiom of choice. Nevertheless, the axiom of choice has no constructive force; a counter-example involving this axiom does not constitute a refutation. On the other hand, for the same reason it is not admissible to refute the axiom by constructing with its help a solution of a problem involving incompatible conditions. The axiom of choice is no argument in favor of Zermelo's well-ordering theorem.

The author discusses a more restrictive system of analysis, as proposed by Borel, which admits only algebraico-differential numbers, such as e and π, as well as the larger system which is used in physics and astronomy. Both are far remote from a total ageometrism, which only admits operative procedures on certain groupings of natural numbers.

E. W. Beth (Amsterdam).

Bouligand, Georges. Sur l'épistémologie de l'analyse classique. C. R. Acad. Sci. Paris **225**, 780–782 (1947).

Referring to an earlier paper [cf. the preceding review], the author first gives a sketch of a calculus of problems. In current language, the word "problem" is used in a more diffuse sense than the one indicated by the author. The so-called problem of Dirichlet, for instance, is a grouping of problems or a composite problem.

The author distinguishes a gradation of sense in the word "true": (1) constructive truth; (2) mixed truth, arising from solution of a composite problem involving pioneering within S_t; (3) truth by consistency, arising from a reductio ad absurdum. The algebraical relations existing between problems establish a kind of pattern which regulates the evolution of S_t.

The theorems regarding the infinite have a twofold aspect: (1) they may be considered, as usual, as statements referring to S_t and involving sets considered as totalities (totalitarian style); (2) statements referring to successive constructive approximations (finitistic style).

As to the admission of nonpredicative notions into S_t, the author observes that there is no permanent guarantee of the consistency of S_t. Though finite, S_t is only vaguely so; it includes litigious zones, which are introduced for the sake of generality, but without sufficient participation from the side of problems. Gradually, under the influence of problems, these zones are organized by means of a readjustment of S_t.

E. W. Beth (Amsterdam).

Bouligand, Georges. Vue d'ensemble sur la mathématique. Rev. Gén. Sci. Pures Appl. N.S. **54**, 151–155 (1947).

San Juan, R. Theory of physical magnitudes and its algebraic foundations. Revista Acad. Ci. Madrid **39**, 11–40, 137–184, 423–461 (1945); **40**, 161–194, 299–336, 495–552 (1946). (Spanish)

Beginning with the second installment the title becomes "Theory of the derived scalar magnitudes and its algebraic foundations."

ALGEBRA

Bailey, W. N. Some identities in combinatory analysis. Proc. London Math. Soc. (2) **49**, 421–435 (1947).

The derivation of the Rogers-Ramanujan and other similar identities is explored to determine a common generalization. The most flexible among several obtained relates a basic ${}_{10}\Phi_{11}$ with a ${}_8\Phi_4$. By suitable limiting processes a large number of identities are obtained, including many of Rogers and those reported by F. J. Dyson [J. London Math. Soc. **18**, 35–39 (1943); these Rev. **5**, 87].

N. A. Hall (Minneapolis, Minn.).

Ljunggren, Wilhelm. An elementary proof of a formula of A. C. Dixon. Norsk Mat. Tidsskr. **29**, 35–38 (1947). (Norwegian)

A well-known result of A. C. Dixon,

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{(n!)^3}$$

[Messenger of Math. **20**, 79–80 (1890)] is derived in an elementary way, by using the identity

$$\sum_{k=0}^{p}\binom{p}{k}\binom{q}{k}\alpha^{p-k}\beta^k=\sum_{k=0}^{p}\binom{p}{k}\binom{q+k}{k}(\alpha-\beta)^{p-k}\beta^k,\qquad q\geqq p.$$

This identity is proved by equating the coefficients of x^p in the developments

$$(\alpha x+\beta)^p(1+x)^q=[x(\alpha-\beta)+\beta(1+x)]^p\cdot(1+x)^q.$$

S. C. van Veen (Delft).

Moran, P. A. P. Rank correlation and permutation distributions. Proc. Cambridge Philos. Soc. **44**, 142–144 (1948).

A formal proof that s, the least number of interchanges of neighbors in a permutation necessary to restore the normal order, is related to Kendall's variable S of rank correlation by $S=\binom{n}{2}-2s$. Since by definition $S=\binom{n}{2}-2i$ [Kendall, The Advanced Theory of Statistics, v. I, London, 1943; Philadelphia, 1944, p. 392; these Rev. **6**, 89], where i is the number of inversions as customarily defined, this amounts to proving $s=i$; but this is an immediate consequence of the facts: (i) a neighbor interchange alters the inversion number by ±1 [Netto, Lehrbuch der Combinatorik, Leipzig, 1901, p. 93] and (ii) the normal order is the only permutation with zero inversions.

J. Riordan.

Plackett, R. L. Cyclic intrablock subgroups and allied designs. Sankhyā **8**, 275–276 (1947).

The author shows that certain designs which were previously constructed by himself and Burman [Biometrika **33**, 305–325 (1946); these Rev. **8**, 44] and some previously constructed by R. A. Fisher [Ann. Eugenics **12**, 283–290 (1945); these Rev. **7**, 107] can be obtained by cyclic permutation of properly chosen initial rows.

H. B. Mann.

Bose, R. C. On a resolvable series of balanced incomplete block designs. Sankhyā **8**, 249–256 (1947).

The author gives a new method of constructing incomplete balanced block designs with the parameters $v=2\lambda+2$,

$c=4\lambda+2$, $r=2\lambda+1$, $k=\lambda+1$, $\lambda=\lambda$, when $2\lambda+1=p^n$, where p is a prime. A nonresolvable solution may always be obtained from the initial blocks $(\infty, x^0, x^2, \cdots, x^{2\lambda-2})$, $(0, x^2, \cdots, x^{2\lambda-2})$, where x is a primitive root of $GF(p^n)$. If λ is odd a resolvable design can be obtained from the initial blocks $(\infty, x^1, x^3, \cdots, x^{2\lambda-1})$, $(0, x^0, x^2, \cdots, x^{2\lambda-2})$.
H. B. Mann (Columbus, Ohio).

Radhakrishna Rao, C. Factorial experiments derivable from combinatorial arrangements of arrays. Suppl. J. Roy. Statist. Soc. **9**, 128–139 (1947).

An ordered set $(a, b, \cdots, k)$ of n symbols $a, b, \cdots, k$ each of which may assume the values $1, \cdots, S$ is called an assembly. A set A of N assemblies is called an array of strength d if each of the S^d possible assemblies corresponding to sets of d symbols occurs in A in any d assigned places an equal number of times. In the case that $N=S^m$ these arrangements have been previously termed hypercubes of strength d by the author [Bull. Calcutta Math. Soc. **38**, 67–78 (1946); these Rev. **8**, 396]. The author discusses new methods for the construction of these arrays in special cases and their application to the construction of symmetrical factorial experiments. In certain cases also asymmetrical designs are obtainable from such arrays. Arrays of strength d may also be used to derive designs for multifactorial experiments which were first introduced by Plackett and Burman [Biometrika **33**, 305–325 (1946); these Rev. **8**, 44]. The main effects and interactions up to order $k<d$ may be estimated from the author's designs when interactions of order greater than d are absent. *H. B. Mann.*

Dorodnov, A. V. On circular lunes quadrable with the use of ruler and compass. Doklady Akad. Nauk SSSR (N.S.) **58**, 965–968 (1947). (Russian)

E. Landau [S.-B. Berlin. Math. Ges. **2**, 1–6 (1903)] and L. Tschakaloff [Math. Z. **30**, 552–559 (1929)] reduced the problem of quadrable lunes to finding pairs of integers m and n, $(m, n)=1$, for which the equation

$$(1)\qquad n\left(\frac{x^m-1}{x-1}\right)^2-mx^{m-n}\left(\frac{x^n-1}{x-1}\right)^2=0$$

is solvable by means of square roots. They investigated the case when m is a prime. T. Claussen [J. Reine Angew. Math. **21**, 375–376 (1840)] had given two quadrable lunes in addition to the three known to Hippocrates of Chios and he made the conjecture that no others exist. N. Tchebotaröw [Math. Z. **39**, 161–175 (1934)] showed this in the case $m-n\equiv 0 \pmod 2$.

The present paper continues the work of Tchebotaröw and completes the solution of the problem by showing that no solutions other than those known to Claussen exist if $m-n\equiv 1 \pmod 2$. The details of the argument are not fully given since the methods employed, the organization of the proof and the notation follow very closely the paper of Tchebotaröw, where the argument depends on p-adic expansion of a root of (1), use of two theorems deduced from work of M. Bauer and of O. Ore, and exclusion of cases by Newton's polygons and other means. *R. Church.*

Carmelo, Gulli. Sulla risoluzione delle equazioni algebriche di 3° e 4° grado. Atti Accad. Ligure **3** (1943), 227–239 (1946).

Solution of the equation of the third degree by means of a linear transformation; brief account of the solution of the equation of the fourth degree by an analogous procedure.
From the author's summary.

Pauli, H. Eingrenzen der Wurzeln von Gleichungen 3. bis 6. Grades. Z. Angew. Math. Mech. **25/27**, 94–95 (1947).

Verfasser gibt für die genannten Funktionen $F(x)$ verschiedene Zerlegungen der Form $f(x)-g(x)=f_1(x)\cdots f_k(x)-g_1(x)\cdots g_m(x)$ an und benutzt die Wurzeln der f_i und g_i und die Vorzeichen in den durch sie bestimmten Intervallen, um Intervalle anzugeben, in welchen möglicherweise Wurzeln von $F(x)$ liegen. Das zugrunde liegende allgemeine und bekannte Prinzip erwähnt er nicht; er übersieht auch, dass man nur für einen einzigen Punkt in einem einzigen Intervall (am einfachsten meist $x=\infty$ oder $x=-\infty$) die Übereinstimmung der Zeichen von $f(x)$ und $g(x)$ festzustellen braucht, da sich dann die der anderen Intervalle von selber ergeben. *E. Bodewig* (Den Haag).

Harlaar, K. Sylvester's determinant. Nieuw Tijdschr. Wiskunde **35**, 174–178 (1947). (Dutch)

A necessary and sufficient condition that two polynomials have a common root is the vanishing of the resultant R (Sylvester's determinant). Sufficiency is proved directly from the special form of R by showing that one solution $y_1, \cdots, y_n$ of the set of linear homogeneous equations corresponding to R has the form $y_k=a^k$. *A. W. Goodman.*

Dieudonné, Jean. Sur la réduction canonique des couples de matrices. Bull. Soc. Math. France **74**, 130–146 (1946).

Considering a finite matrix over a commutative field K as an endomorphism of a vector space E, the author develops the theory of elementary divisors and applies it to the reduction to canonical form of pairs of matrices, in particular to the case where one is symmetric and the other skew. The method throws new light on certain properties of the elementary divisors of such pairs. Most of the results except the actual canonical form are valid in any field not necessarily commutative. Let f_p be the pth iterate of the endomorphism f and let r be the least integer such that $f_{r+1}(E)=f_r(E)$ so that $E=f_r(E)\dot{+}B_r$. Then B_r is a direct sum of subspaces $f_k(C_p)$, $1\leqq k<p$, $1\leqq p\leqq r$, where C_p is of dimension h_p. Let Γ be the group of automorphisms of E commutative with f. There is a composition series

$$\Gamma, \Gamma_r, \Gamma_r'', \Gamma_r', \Gamma_{r-1}, \Gamma_{r-1}'', \Gamma_{r-1}', \cdots, \Gamma_2, \Gamma_2'', \Gamma_2', \{e\},$$

such that Γ/Γ_r is isomorphic with the linear group $GL(h_r)$; Γ_p/Γ_p'' is isomorphic with $GL(h_{p-1})$ for $2\leqq p\leqq r$; Γ_p''/Γ_p' is Abelian, the direct sum of $h_{p-1}(h_p+\cdots+h_r)$ groups isomorphic with the addition group of the field K; and Γ_p'/Γ_{p-1} is Abelian, the direct sum of $(h_{p-1}+\cdots+h_r)(h_p+\cdots+h_r)$ groups isomorphic with K. *C. C. MacDuffee.*

Dieudonné, Jean. Compléments à trois articles antérieurs. II. Bull. Soc. Math. France **74**, 65–66 (1946).

The author removes restrictions imposed in a previous paper [same Bull. **71**, 27–45 (1943); these Rev. **7**, 3] and shows that certain of his results on determinants are valid in any noncommutative field. *N. H. McCoy.*

★**Rademacher, Hans. On a theorem of Frobenius.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 301–305. Interscience Publishers, Inc., New York, 1948. $5.50.

Let $\mathfrak{S}$ be a set of commutative matrices $A, B, C, \cdots$ of order n with elements in an algebraically closed field Φ. For each characteristic root α of A there exists a characteristic vector $\mathfrak{z}$ of all matrices of $\mathfrak{S}$, and it determines a correspondence between the characteristic roots $\alpha, \beta, \gamma, \cdots$

of $A, B, C, \cdots$ to which it belongs. This leads to a proof of the theorem of Frobenius [S.-B. Preuss. Akad. Wiss. 1896, 601–614] that the characteristic roots $\alpha_1, \alpha_2, \cdots$ of A; $\beta_1, \beta_2, \cdots$ of B, etc., can be so arranged that the characteristic roots of $f(A, B, \cdots)$ are $f(\alpha_i, \beta_i, \cdots)$ for every rational function f. *C. C. MacDuffee* (Madison, Wis.).

Lepage, Th.-H. Sur un théorème de Kronecker relatif aux sous-déterminants d'une matrice symétrique. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 288–299 (1947).

Lepage, Th.-H. Sur un théorème de Kronecker relatif aux sous-déterminants d'une matrice symétrique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 527–541 (1947).

Let Ω_p, Γ denote two alternating forms (skew-symmetric tensors) of respective degrees p and 2 defined in a linear space of $N\geqq 2n$ dimensions. All multiplication is outer in the sense of Grassmann. Assume $\Gamma=\xi_1\eta_1+\cdots+\xi_n\eta_n$ and consider the n linear forms $\omega_i=\eta_i-p_{i1}\xi_1-\cdots-p_{in}\xi_n$, $i=1, \cdots, n$. It is proved that a necessary and sufficient condition in order that Ω_p have Γ as a factor is that $\Omega_p\omega_1\cdots\omega_n$ vanish for every symmetric matrix $P=(p_{ij})$. Incidentally to this proof, the author establishes a division algorithm: $\Omega_p=\theta_{p-2}\Gamma+\rho_p$, $0\leqq p\leqq 2n$, where $\rho_p\Gamma^{n-p+1}=0$. The quotient θ_{p-2} and the residue ρ_p are unique. Every module $M(p, \Gamma)$, $1\leqq p\leqq n$, of residues ρ_p of Γ has a basis composed of completely decomposable forms.

As an application the author easily obtains the theorem of Radon [Monatsh. Math. Phys. **48**, 198–204 (1939); these Rev. **1**, 98] that, if A and B are n by n matrices and if the n by $2n$ array (A, B) is of rank r, there exists a symmetric matrix P such that $A+BP$ is of rank r. He also obtains the Kronecker-Runge theorem [S.-B. Preuss. Akad. Wiss. 1882, 821–824; J. Reine Angew. Math. **93**, 319–327 (1882)] on the linear relations among the subdeterminants of a symmetric matrix. *C. C. MacDuffee* (Madison, Wis.).

Madhava Rao, B. S. Generalised algebra of elementary particles. Proc. Indian Acad. Sci., Sect. A. **26**, 221–233 (1947).

The equation $\partial_\mu\beta_\mu\psi+\kappa\psi=0$ has been generalized by Kemmer for spin eigenvalues $\pm 1, 0$ to the case in which the β_μ consist of an arbitrary number of elements s, and by Bhabha, for $s=4$, to describe particles of spin eigenvalues $r-u$ ($u=0, \cdots, 2r$). In this paper the generalization to both arbitrary r and s is carried out, both for r a positive integer and half an odd integer. If certain conditions introduced by Bhabha are generalized to arbitrary s, the representations of the β algebra are then those of the real orthogonal group in $s+1$ dimensions. For the meson case ($r=1$) an alternative method of deriving Kemmer's results is obtained, and equally simple results emerge for the generalized s-element algebra corresponding to $r=3/2$.

H. C. Corben (Pittsburgh, Pa.)

Abstract Algebra

Riguet, Jacques. Produit tensoriel de lattices. C. R. Acad. Sci. Paris **226**, 40–41 (1948).

In a manner analogous to Whitney's treatment of groups [Duke Math. J. **4**, 495–528 (1938)], the author defines two forms of product of lattices. Without proof, simple properties are stated, notably that the product is distributive even when the given lattices are not. *P. M. Whitman.*

Riguet, Jacques. Produit tensoriel de lattices. C. R. Acad. Sci. Paris **226**, 143–146 (1948).

Continuing the paper reviewed above, the author defines the product of tensors derived from lattices. If the lattices are Boolean algebras, a contraction of a tensor is defined and some properties and analogies to ordinary tensor analysis stated. Finally, a connection with the theory of relations is indicated. *P. M. Whitman* (Silver Spring, Md.).

Monteiro, António. Sur l'arithmétique des filtres premiers. C. R. Acad. Sci. Paris **225**, 846–848 (1947).

The arithmetic of ideals (dual ideals, or filters) of a lattice L is said to be extremal if all the prime ideals (prime dual ideals) of L are maximal. If the two nonequivalent conditions (a) every ideal is the intersection of maximal ideals, (b) every dual ideal is the intersection of maximal dual ideals, are satisfied, the arithmetic of L is said to be maximal. The author considers the relationship between extremal and maximal arithmetics of distributive lattices and states the following partial results. (1) A necessary and sufficient condition that the arithmetic of ideals (dual ideals) of a distributive lattice be extremal is that it be relatively complemented. (2) A conditionally complete distributive lattice has a maximal arithmetic of ideals and dual ideals if it is relatively complemented. In particular, a complete distributive lattice with a maximal arithmetic of ideals is a Boolean algebra. (3) An inf-complemented distributive lattice satisfying (b) above is relatively complemented. (A lattice L is called inf-complemented if for each $x\epsilon L$ and $a\leqq x\leqq b$ there exists an $x_0\leqq b$ such that (i) $x\cap x_0=a$; (ii) if $y\leqq b$ and $x\cap y=a$, then $y\leqq x_0$.) Whether there exist distributive lattices having maximal arithmetics which are not extremal is not known. *W. D. Duthie* (Annapolis, Md.).

Zariski, Oscar. Generalized semi-local rings. Summa Brasil. Math. **1**, no. 8, 169–195 (1946).

Let R be a Noetherian ring. If $\mathfrak{m}$ is an ideal in R, then R is said to be $\mathfrak{m}$-adic if $\cap_{i=1}^\infty\mathfrak{m}^i=(0)$; R may then be topologized by taking the $\mathfrak{m}^i$ as neighborhoods of zero. The $\mathfrak{m}$-adic ring R is said to be semi-local if $\mathfrak{m}\neq(0)$ and if all ideals are closed. It is proved that the completion R^* of an $\mathfrak{m}$-adic ring R is Noetherian, $R^*\mathfrak{m}$-adic, and semi-local. Some equivalent formulations of these definitions are given. The relationship between the ideal theory of R and R^* is discussed, and several formulas for the closure $\bar{\mathfrak{B}}$ of an ideal $\mathfrak{B}$ in R are given; for example, $\bar{\mathfrak{B}}=R^*\cdot\mathfrak{B}\cap R$. If the intersection $\mathfrak{R}$ of all the maximal ideals of R is not equal to (0), then it is shown that R is $\mathfrak{R}$-adic and semi-local. The special case where $\mathfrak{R}$ is the intersection of a finite number of ideals gives the semi-local rings of Chevalley; the case where $\mathfrak{R}$ is maximal gives a local ring.

The main theorem states that a Noetherian integral domain R is analytically irreducible (i.e., the $\mathfrak{R}$-adic completion R^* has no zero divisors) if, at each maximal ideal $\mathfrak{p}$, R is locally irreducible (i.e., the $\mathfrak{p}$-adic completion $R^*\mathfrak{p}$ has no zero divisors) and if $\mathfrak{R}$ is connected (i.e., $\mathfrak{R}$ is not the intersection of two proper co-maximal ideals). For the proof there is defined a certain natural homomorphism of R^* into $R^*\mathfrak{p}$, and it is shown that a sufficient condition for the vanishing of an element x^* in R^* is the vanishing of its analytical element (i.e., its map in $R^*\mathfrak{p}$) for each maximal ideal $\mathfrak{p}$. Under the hypotheses of the main theorem, it can

then be proved that the mapping of R^* into $R^*\mathfrak{p}$ is actually an isomorphism and thus R^* is an integral domain.

The material of this paper is to be applied to the semi-local theory of algebraic varieties. *I. S. Cohen.*

Goldman, Oscar. Addition to my note on semi-simple rings. Bull. Amer. Math. Soc. **53**, 956 (1947).

In Ergänzung zu seiner früheren Arbeit [Bull. Amer. Math. Soc. **52**, 1021–1027 (1946); diese Rev. **8**, 433] bemerkt Verfasser, dass die von Chevalley gegebene Definition: Das Radikal eines Schiefringes A ist der Durchschnitt aller Annullatoren einfacher A-Moduln, gleichwertig mit der von N. Jacobson [Amer. J. Math. **67**, 300–320 (1945); diese Rev. **7**, 2] gegebenen ist, sodass Satz I und II seiner Arbeit überflüssig werden. *H. Zassenhaus* (Hamburg).

***Artin, Emil. Linear mappings and the existence of a normal basis.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 1–5. Interscience Publishers, Inc., New York, 1948. $5.50.

L'auteur donne une nouvelle démonstration de l'existence d'une base normale d'une extension algébrique, finie, galoisienne et séparable K d'un corps k, autre que les champs de Galois. Il se base sur un théorème, qui peut s'énoncer essentiellement comme suit: (1) $K(X)=K(x_1, \cdots, x_n)$ étant un espace vectoriel d'un rang fini n sur K et $f(X)=f(x_1, \cdots, x_n)$ étant un polynome dans K par rapport aux coordonnées $x_1, \cdots, x_n$ du point X de $K(X)$, $f(X)$ ne peut être le polynome du degré total le plus bas, tel que l'ensemble des solutions de $f(X)=0$ contienne un module M, que si l'on a identiquement $f(X+Y)=f(X)+f(Y)$; (2) si $f(X)$ est un polynome tel que $f(X+Y)=f(X)+f(Y)$, il est de premier degré quand la caractéristique de K est nulle, et est une somme de polynomes $f_i(x_i)$ d'une variable x_i de la forme $f_i(x_i)=\sum a_{i\sigma}x_i^{p^\sigma}$ (p-polynomes de Ore) quand la caractéristique de K est p.

On démontre (1) en remarquant que l'ensemble des solutions en X de $H(X, Y)=f(X+Y)-f(X)-f(Y)=0$ contient M pour tout $Y\epsilon M$, quoique $H(X, Y)$ est de degré total en X plus petit que le degré φ de $f(X)$, d'où, pour tout $Y\epsilon M$, on a $H(X, Y)=0$, identiquement en X. Par suite les coefficients de ce polynome de X, qui sont les polynomes en Y de degré plus petit que φ, ont M comme partie de l'ensemble des solutions, donc sont identiquement nuls. Il est évident que $H(X, Y)=0$ entraîne que $f(X)$ est la somme de polynomes $f_i(x_i)$ de la même forme d'une seule variable. Les parties homogènes $a_i^{(s)}x_i^s$ de $f_i(x_i)$ doivent aussi être de la même forme, d'où il résulte facilement que si $a_i^{(s)}\neq 0$, s est une puissance de la caractéristique p de K. La partie (2) et, pour $n=1$, la partie (1) de ce théorème sont déjà connues et semblent avoir été démontrées pour la première fois par Ore [Trans. Amer. Math. Soc. **35**, 559–584 (1933)]. Une démonstration identique à celle de l'auteur a été donnée par Krasner [Mathematica, Cluj **13**, 72–191 (1937), § 6].

Soient $\lambda_1, \cdots, \lambda_n$ des homomorphismes dans K d'un groupe additif A et $\Lambda(a)$, $a\epsilon A$, le point $x_i=\lambda_i(a)$ de $K(X)$; $\lambda_1, \cdots, \lambda_n$ seront dits algébriquement dépendants s'il existe un polynome $f(X)$ de X dans K tel qu'on ait $f(\Lambda(a))=0$ pour tout $a\epsilon A$. Ils seront dits linéairement dépendants s'il existe un tel $f(X)$ linéaire, c'est-à-dire tel que $f(X+Y)=f(X)+f(Y)$. Comme $\Lambda(A)$ est un module, la dépendance algébrique des λ_i entraîne leur dépendance linéaire. D'autre part, il est connu que n homomorphismes distincts $\mu_1, \cdots, \mu_n$ d'un groupe multiplicatif dans K ne peuvent pas être liés par une relation de premier degré. En particulier, si $\lambda_1, \cdots, \lambda_n$ forment un groupe G d'automorphismes de K, ils sont algébriquement indépendants, car sinon ils seraient, en tant que homomorphismes additifs, linéairement dépendants. Ainsi, si K est de caractéristique 0, ils satisferraient, en tant que homomorphismes multiplicatifs, à une relation de premier degré, ce qui est absurde. Dans le cas de caractéristique $p\neq 0$, on arrive à une contradiction d'une manière un peu plus compliquée, quand K n'est pas un champ de Galois.

Soit u_{λ_i} une indéterminée associée à λ_i, $(u_{\lambda_i\lambda_j^{-1}})$ la matrice du groupe G et U son déterminant; on sait que U n'est pas identiquement nul en u_{λ_i}. Si $|\lambda_i\lambda_j^{-1}(a)|$ était nul pour tout $a\epsilon K$, donc $\lambda_1, \cdots, \lambda_n$ auraient été algébriquement dépendants, ce qui est impossible. Il existe donc un $a\epsilon K$ tel que $|\lambda_i\lambda_j^{-1}(a)|\neq 0$, ce qui suffit pour assurer l'existence d'une base normale de K par rapport au corps d'invariants k de G. Donc $\lambda_1, \cdots, \lambda_n$ sont aussi multiplicativement indépendants car de $\prod\lambda_i(a)^{n_i}=1$, on peut tirer une relation algébrique entre les λ_i. *M. Krasner* (Paris).

Arocena, Antonio. Galois theory. Revista Acad. Ci. Madrid **38**, 11–54, 174–194, 331–348 (1944). (Spanish)
Expository article.

Schwarz, Štefan. On the extension of the Jordan-Kronecker's "principle of reduction" for inseparable polynomials. Časopis Pěst. Mat. Fys. **72**, 61–64 (1947). (English. Czech summary)

The author extends results of a previous paper [same Časopis **71**, 17–20 (1946); these Rev. **8**, 500] to the inseparable case. Let f and g be two irreducible polynomials over a field P and let each be factored into a product of powers of irreducible factors in the field obtained by adjoining to P a zero of the other. Then there is a unique one-to-one correspondence $f_i\sim g_i$ between the irreducible factors of f and of g, such that, for each i, f_i and g_i not only satisfy all the relations given in the paper cited above, but also have the same exponent in the factorization of f and of g, respectively.

These facts are easy consequences of the theory of commutative algebras. F. K. Schmidt has also given proofs of them [S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1925**, no. 5, 19–26]. *G. Whaples* (Bloomington, Ind.).

Levi, F. W. Pairs of inverse modules in a skewfield. Bull. Amer. Math. Soc. **53**, 1177–1182 (1947).

The concept of "pairs of inverse modules" was introduced by the author [J. Indian Math. Soc. N.S. **3**, 295–306 (1939); these Rev. **1**, 198]: two sub-modules of a skew-field Σ are said to be inverses of one another if the nonzero elements of one are the inverses of those of the other. A module which possesses an inverse module will be called a J-module, and a self-inverse module which contains the unit element of Σ will be called an S-module. The author has shown [loc. cit.] that in a commutative field S-modules which are not subfields can exist only for characteristic 2. In the noncommutative case the simplest example of a nontrivial S-module is given by the module generated by $[1, i, j]$ in the quaternion field with rational coefficients.

In the present paper the author proves the following theorems. (1) A submodule of Σ is a J-module if and only if it contains with the elements a and $b\neq 0$ also $ab^{-1}a$. (From this it follows that the J-modules of Σ form a lattice with set-inclusion as order-relation.) (2) If an S-module contains the elements a, b and ab, then it contains the whole skew-field generated by a and b. (3) Every finite S-module is a Galois field. *K. A. Hirsch.*

Ballieu, Robert. Anneaux finis; systèmes hypercomplexes de rang trois sur un corps commutatif. Ann. Soc. Sci. Bruxelles. Sér. I. 61, 222–227 (1947).

The author continues his study of finite rings [same Ann. Sér. I. 61, 117–126 (1947); these Rev. 8, 499] by finding all rings of order p^3. For those whose additive group is of type (p, p, p) this means finding all algebras of order 3 over $GF(p)$. More generally, the author finds the algebras of order 3 over an arbitrary field. [Bibliographical note. All algebras of order 3 were listed by Scorza [Atti Acad. Sci. Fis. Mat. Napoli (2) 20, no. 13 (1935)]. The algebras of order 4 were also found by Scorza [ibid., no. 14 (1935)], and, in unpublished work, by Albert and Wedderburn [Albert, Structure of Algebras, Amer. Math. Soc. Colloquium Publ., v. 24, New York, 1939, p. 172; these Rev. 1, 99]. Nilpotent algebras of order not exceeding 6 were classified by Boyce [Chicago dissertation, 1938].]

I. Kaplansky (Chicago, Ill.).

Hochschild, G. Cohomology and representations of associative algebras. Duke Math. J. 14, 921–948 (1947).

In two previous papers [Ann. of Math. (2) 46, 58–67 (1945); 47, 568–579 (1946); these Rev. 6, 114; 8, 64] the author has introduced the concept of cohomology groups of associative algebras and has started their investigation. The main objective of the present contribution is to find an interpretation of the third cohomology group $H^3(A, N)$ of the F-algebra A over the two-sided A-module N. This interpretation is obtained in the following fashion. If K is an algebra over the field F, and if $m=(u(x), v(x))$ is a pair of linear K to K transformations such that $xu(y)=v(x)y$, $u(xy)=u(x)y$, $v(xy)=xv(y)$ for x, y in K, then m is a multiplication on K; and m is an inner multiplication in K if there exists an element k in K such that $m=(kx, xk)$. Defining sum and product of multiplications by the rules $(u', v')+(u'', v'')=(u'+u'', v'+v'')$, $(u', v')(u'', v'')=(u'u'', v''v')$, $a(u, v)=(au, av)$ for a in F, the system $M(K)$ of all the multiplications in K is turned into an algebra over F; the inner multiplications form a two-sided ideal $I(K)$ in $M(K)$.

A linear mapping r of the F-algebra A into $M(K)$ is regular if the right components of all the multiplications $r(A)$ commute with all the left components of multiplications in $r(A)$. If the product f of the regular linear mapping r by the natural homomorphism of $M(K)$ upon $M(K)/I(K)$ happens to be a homomorphism of the algebra A over F into the algebra $M(K)/I(K)$ over F, then the pair $[f, K]$ is a representation of A in K; and r is compatible with f. If N is the annihilator of the algebra K, and if $r(a)=[(u(x), v(x))]$, then N is made into a two-sided A-module by the rules $an=u(n)$, $na=v(n)$ for a in A and n in N. Since the structure of this module depends on f alone, this A-module N is termed the nucleus of the representation $[f, K]$. If a, b are in A, then $r(a)r(b)-r(ab)$ is an inner multiplication effected by some element $h(a, b)$ in K. The coboundaries of these "hindrances" $h(a, b)$ constitute a definite element in $H^3(A, N)$, the "obstruction" of the representation $[f, K]$. It is shown that the obstruction of the representation $[f, K]$ is 0 if, and only if, $[f, K]$ has been derived (in an obvious fashion) from an extension of A. This leads to a natural definition of an additive F-group of classes of similar representations with fixed nucleus N; this group is mapped isomorphically upon $H^3(A, N)$ by mapping representations on their obstructions.

If F is a field, then $K_n(F)$ is the class of all algebras A over F such that $H^n(A, N)=0$ for every A-module N. The author shows that $K_2(F)<K_3(F)$, that nilpotent algebras do not belong to $K_3(F)$, that semisimple inseparable algebras do not belong to any $K_n(F)$ and that the group rings of finite groups whose orders are multiples of the characteristic of F do not belong to any $K_n(F)$. *R. Baer.*

THEORY OF GROUPS

***Kuroš, A. G. Teoriya Grupp. [Theory of Groups].** OGIZ, Moscow-Leningrad, 1944. 372 pp. (Russian)

This book is a fairly complete and very clear account of the theory of abstract groups. It begins at the very beginning of the subject and its leisurely style that gives ample motivations and historical references should make the book particularly valuable as a text book. On the other hand the author has gone sufficiently deeply into the theory to make the book also useful as a reference work for active workers in the field. There are a number of topics treated here that, to the reviewer's knowledge, do not appear in any of the other treatises. Particularly noteworthy is the Prufer-Ulm-Zippin theory of primary Abelian groups, the theory of infinite solvable and infinite p-groups, the theory of free groups and free products, and the study of relations between lattice theory and group theory. A striking feature of the book is its complete emancipation from the finite group point of view; finite groups are relegated to the role of a special case throughout.

The following is a list of the chapter titles: (I) Definition of a group, (II) Subgroups and normal divisors, (III) Generating relations, (IV) Automorphisms and endomorphisms, series of subgroups, (V) Abelian groups with a finite number of generators, (VI) Direct products, extensions of groups, (VII) Solvable groups, p-groups, special groups, (VIII) Primary Abelian groups, (IX) Abelian groups without torsion and mixed groups, (X) Free groups and free products, (XI) Structures (lattices) and groups.

Many unsolved problems, some classical and some new, are mentioned throughout the text. There is an extensive bibliography. *N. Jacobson* (New Haven, Conn.).

Tang, Tsao-Chen. Some new fundamental characteristics of groups. Wu-Han Univ. J. Sci. 8, no. 1, 0.1–0.15 (1942). (Chinese)

Let G be a group, with elements $a, b, c, \cdots$, and let x denote the group operation. Given any element p of the group, define a new operation x_p by $ax_pb=axp^{-1}xb$. Under this operation, the elements of G form another group G_p, whose unit element is p. Then G_p is isomorphic to G under the transformation $a\rightarrow axp$. The set of all such isomorphisms forms another group in an obvious fashion. Various other elementary observations are made about such a structure.

E. G. Begle (New Haven, Conn.).

Tseng, Hsien-Chang. Theory of permutation groups. Wu-Han Univ. J. Sci. 8, no. 1, 2.1–2.12 (1942). (Chinese) Expository article.

Rankin, R. A. A campanological problem in group theory. Proc. Cambridge Philos. Soc. 44, 17–25 (1948).

Although he goes deeper for the sake of completeness, the author is mainly concerned with the question as to

whether the elements of a group generated by two elements α and β can all be arranged in such a cyclic order that each is derived from its predecessor by multiplication on the left by either α or β. For instance, the symmetric group of degree 3 has this property for two transpositions α and β, since its 6 elements are 1, α, $\beta\alpha$, $\alpha\beta\alpha$, $\beta\alpha\beta\alpha$, $\alpha\beta\alpha\beta\alpha$. Let l, m, n and l_1, m_1, n_1 be the orders and indices of the cyclic subgroups generated by α, β, $\alpha^{-1}\beta$ (so that the order of the whole group is $ll_1=mm_1=nn_1$). Then if n is odd, a necessary condition for the above arrangement is that l_1 and m_1 be odd.

The object of the campanological "exercise" is to ring as many changes as possible on a given set of $q(\geqq 7)$ bells. This is done by using a few simple rules to derive each successive permutation. These rules amount to multiplication by two or three special permutations called "plain leads," "bobs," etc. Methods based on various choices of these special permutations have picturesque names such as "Grandsire triples"; and much trouble used to be taken to prove that in each case it is impossible to ring the complete peal of $q!$ changes using plain leads (α) and bobs (β) alone. The new theorem enables us to deduce this impossibility from the simple observation that, in the alternating group of degree $q-1$ generated by α and β, the period of $\alpha^{-1}\beta$ is odd. (This suffices, since l_1 and m_1 could not possibly be odd in such a case.) *H. S. M. Coxeter* (Toronto, Ont.).

Hamill, C. M. On a finite group of order 576. Proc. Cambridge Philos. Soc. **44**, 26–36 (1948).

The author obtains the sixteen classes of conjugates of a certain primitive finite group G of order 576, by describing G as a group of collineations which leave invariant a configuration of two associated triads of tetrahedra. Each vertex and opposite face are said to be a pole and its polar plane. The corresponding harmonic inversion is called a projection, and the 24 projections thus obtained are shown to generate the group. The projections form two conjugate sets of 12 each, called type II. Each operation of G is the product of at most four projections. Products of projections from two vertices of the same tetrahedron give 9 elements of period 2 forming a single conjugate set called type V. Products of two distinct elements of type V give elements of the same type and also 6 other elements of period 2 forming a conjugate set called type IX. It might have simplified the derivations in the paper had the author pointed out that these 15 elements of period 2, together with identity, form an invariant subgroup of order 16, leaving each of the six tetrahedra fixed, and that the quotient group is the direct product of two symmetric groups of degree 3, which represent the permutations of the tetrahedra within the two triads. The conjugate sets for the quotient group could then be split into conjugate sets for G.
J. S. Frame (East Lansing, Mich.).

Kaloujnine, Léo. Sur les groupes abéliens primaires sans éléments de hauteur infinie. C. R. Acad. Sci. Paris **225**, 713–715 (1947).

An f-group is a primary Abelian group without elements of infinite height. If G is any additive f-group, the chain of subgroups p^mG, $m=0, 1, 2, \cdots$, defines a natural topology in G, which is discrete if and only if the elements of G have bounded orders. A subgroup H of G is said to be a subordinate subgroup of G ("sous-groupe servant") when $p^mH=H\cap p^mG$ for all m. The following theorem is announced. Every f-group G contains a subgroup H which is (i) subordinate in G, (ii) everywhere dense in G in the sense of the natural topology in G, (iii) decomposable into a direct sum of cyclic subgroups. This subgroup H is uniquely determined up to an isomorphism by the statements (i), (ii), (iii). Let $\bar{G}$ be the completion of G in the natural topology (assumed nondiscrete) and let G^* be the maximal periodic subgroup of G; then G^* is called the quasi-completion of G, and G is quasi-complete if $G=G^*$. The following results are stated. For any G, G^* is quasi-complete and G is subordinate in G^*. No decomposable G is quasi-complete, but every quasi-complete G is the quasi-completion of a decomposable subgroup H. *S. A. Jennings* (Vancouver, B. C.).

Lesieur, Léonce. Sur la multiplication des fonctions caractéristiques de Schur. C. R. Acad. Sci. Paris **225**, 848–850 (1947).

The paper describes briefly a method of multiplying an S-function (Schur's characteristic function) by a monomial symmetric function, and thence obtains a method of multiplying two S-functions. The results, however, were worked out in careful detail by F. D. Murnaghan in 1938 [Amer. J. Math. **60**, 44–65]. *D. E. Littlewood.*

Corson, E. M. Note on the Dirac character operators. Physical Rev. (2) **73**, 57–60 (1948).

The Dirac character operators are evaluated for the classes of the symmetric group consisting of the triad, double interchange, tetrad, and pentad, and the corresponding group-theoretical primitive characters are listed for comparison. It is also shown that the evaluation of these and similar character operators is all that is required for the solution of the standard molecular problems in the spirit of Dirac's original program which avoids appeal to formal group theory. *Author's summary.*

Brauer, Richard. Applications of induced characters. Amer. J. Math. **69**, 709–716 (1947).

L'auteur applique son théorème: tout caractère χ d'un groupe G d'ordre fini g est une combinaison linéaire à coefficients entiers rationnels des caractères ω^* induits par les caractères ω de premier degré de sous-groupes H de G (appellés sous-groupes élémentaires) qui sont des produits directs $A_p\times B$ d'un groupe A_p dont l'ordre soit une puissance de quelque premier p et d'un groupe cyclique B d'ordre premier à p [Ann. of Math. (2) **48**, 502–514 (1947); ces Rev. **8**, 503], à l'étude de plusieurs problèmes de la théorie des représentations de groupes. D'abord, il donne une démonstration simplifiée de son théorème antérieur [Amer. J. Math. **67**, 461–471 (1945); ces Rev. **7**, 238] que tout représentation de G peut s'écrire dans le corps des racines g-ièmes de l'unité, en montrant, d'ailleurs, qu'on peut remplacer g par le plus petit commun multiple n des ordres des éléments de G. En effet, tout représentation de premier degré ω d'un groupe élémentaire H, et aussi son induite ω^*, peut s'écrire dans ce corps K. D'autre part, un théorème de Schur montre que si l'on exprime quelque ω^* de la forme précédente sous la forme $\omega^*=\sum a_\chi^{(\omega^*)}\chi$, où χ parcourt les caractères irréductibles de G, l'indice de Schur de la représentation r_χ du caractère χ par rapport à K (c'est-à-dire le degré par rapport à K de son plus petit surcorps où l'on peut l'écrire) divise $a_\chi^{(\omega^*)}$. Quel que soit le premier p, le théorème cité de l'auteur assure l'existence d'un ω^* tel que $a_\chi^{(\omega^*)}\not\equiv 0$ (mod p). Donc, l'indice de Schur de r_χ par rapport à K n'est divisible par aucun p, donc est 1. L'auteur précise, ensuite, les relations entre les indices de Schur des représentations de G par rapport au corps K_0 engendré par les caractères des éléments de G et la structure de G.

Soient h l'ordre du sous-groupe H de G; $C_1, \cdots, C_e$ les classes d'éléments conjugués de G; $L_1, \cdots, L_t$ celles d'éléments conjugués de H; g_i, h_j les nombres d'éléments des C_i, L_j. Puisque, pour un $a\epsilon G$, on a $\omega^*(a)=h^{-1}\sum\omega(xax^{-1})$, où x parcourt G, on a $\omega^*(C_i)=(g/hg_i)\sum h_j\omega(L_j)$, où L_j parcourt les classes contenues dans C_i. Ainsi, pour construire tous les caractères ω^* induits par les caractères ω de premier degré des groupes élémentaires, il suffit, quand on sait déjà combien il y a de C_i et quels sont leurs nombres d'éléments g_i, de connaître pour chaque sous-groupe élémentaire H, combien il y a de classes L_j d'éléments conjugués de H, quels sont leurs nombres d'éléments h_j, dans quel $C_{i(j)}$ est contenue chaque L_j et, pour chaque caractère linéaire ω de H, sa valeur $\omega(L_j)$ pour chaque L_j; cette dernière donnée est fournie, par exemple, par la table de multiplication des L_j, qui permet de déterminer toutes les subdivisions des L_j en classes qui correspondent aux groupes quotients cycliques de H par ses sous-groupes invariants. Les ω_ρ^* de la forme précédente étant construits et $m_{\rho\sigma}$ étant $g^{-1}\sum g_i\omega_\rho^*(C_i)\bar{\omega}_\sigma^*(C_i)$, où $\bar{\omega}_\sigma^*$ est le complexe conjugué de ω_σ^*, en vertu des relations d'orthogonalité, une combinaison linéaire $\chi=\sum x_\rho\omega_\rho^*$ des ω_ρ^* à coefficients entiers est un caractère irréductible de G si, et seulement si, $\sum m_{\rho\sigma}x_\rho x_\sigma=1$ et $\sum x_\rho\omega_\rho^*(1)>0$, où 1 est l'unité de G; il n'y a qu'un nombre fini de solutions non-équivalentes, c'est-à-dire donnant des χ différents, de cette équation diophantienne, et on peut les déterminer par un nombre fini d'opérations rationnelles.

Ensuite, l'auteur prouve que tout caractère ϕ d'une constituante indécomposable de la représentation régulière modulaire R_p (mod p) de G est une combinaison linéaire à coefficients entiers des caractères ω_ρ^* induits par les caractères ω_ρ de premier degré des groupes élémentaires H de G d'ordre premier à p, ce qui est l'analogue, pour les caractères modulaires, du théorème cité de l'auteur. Comme dans la démonstration de ce dernier théorème il suffit de prouver que, pour tout idéal primaire q^t du corps K, où l'on peut écrire toutes les représentations, la vérification, pour les q-entiers x_i de K et pour tout ω_ρ^*, de $\sum x_i\omega_\rho^*(C_i)\equiv 0$ (mod q^t) entraîne celle de $\sum x_i\phi(C_i)\equiv 0$ (mod q^t). Si q ne divise pas p, la démonstration du travail cité reste valable. Pour $q|p$, l'auteur donne une autre démonstration, basée sur sa théorie des caractères modulaires.

La connaissance des caractères ω_ρ de premier degré des groupes élémentaires H d'ordre premier à p permet de construire une base minimale du module des caractères modulaires ϕ (mod p), mais non sa base spéciale formée par les caractères ϕ_r des composantes indécomposables de R_p. Toutefois, si $\psi_1, \psi_2, \cdots, \psi_w$ est une base de ce module, et si $q_{\alpha\beta}=g^{-1}\sum g_i\psi_\alpha(C_i)\bar{\psi}_\beta(C_i)$, l'auteur montre que la matrice $(q_{\alpha\beta})$ de la forme quadratique $\sum q_{\alpha\beta}x_\alpha x_\beta$ est équivalente à la matrice de Cartan $(c_{\alpha\beta})$ de G pour p et que le nombre des représentations ordinaires irréductibles de G, dont le degré est divisible par la contribution p^a de p dans g, est égal à celui des représentations de 1 par la forme $\sum q_{\alpha\beta}x_\alpha x_\beta$, où x_α, x_β sont des entiers rationnels, en comptant $x_1, \cdots, x_w$ et $-x_1, \cdots, -x_w$ comme une même representation.

M. Krasner (Paris).

✱Fenchel, W., and Nielsen, J. On discontinuous groups of isometric transformations of the non-Euclidean plane. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 117–128. Interscience Publishers, Inc., New York, 1948. \$5.50.

Let E be the unit circumference in the complex plane and let D be the interior of E. Hyperbolic geometry is represented in this unit circle in the usual way with the points at infinity being on E. Groups of isometric transformations are considered and a point or line is called invariant under G if it is fixed under all the elements of G. A group G is said to be discontinuous in a point x if x is not an accumulation point of the orbit Gx. If G is discontinuous in one point of D, then it is discontinuous in the whole of D. The authors prove the following theorem. If G is a group of isometric transformations of the hyperbolic plane which leaves invariant no point and no line, then a necessary and sufficient condition for G to be discontinuous in D is that the centers of the rotations in G, if any, do not accumulate in D.

D. Montgomery (New Haven, Conn.).

NUMBER THEORY

Ryde, Folke. Tafel und Nomogramm der monotonen, nichtwachsenden Kettenbrüche. Ark. Mat. Astr. Fys. **34A**, no. 11, 13 pp. (1 plate) (1947).

This paper contains a table of values for $Nm(r)$, where $Nm(r)$ is the number of developments of a rational number r as monotonic nonincreasing continued fractions. The computations are based on results obtained in previous papers [same Ark. **31A**, no. 19 (1945); **31B**, no. 12 (1945); these Rev. **8**, 5]. The table lists all $Nm(r)$ for r in the form $(p-q)/q$, where $p\leqq 50$ and $q\leqq p-1$ are positive integers. The representations themselves are also listed except that, for r an integer, they are restricted to the cases $r<15$. The paper contains a nomogram which is useful for checking some of the calculations in the table.

W. H. Gage.

Ryde, Folke. Eine Produktdarstellung der monotonen, nichtwachsenden Kettenbrüche. Ark. Mat. Astr. Fys. **34A**, no. 16, 16 pp. (1947).

Let $P_n=x_0\prod_{i=1}^{n}\{1+(-1)^{i-1}/x_i\}$. After expressing P_n as a continued fraction of $n+1$ terms, the author determines conditions under which a given continued fraction with positive numerators and denominators is equivalent to a product of the form P_n. Necessary and sufficient conditions that a rational number r may be developed in a monotonic nonincreasing continued fraction [same Ark. **31A**, no. 19 (1945); these Rev. **8**, 5] of $n+1$ terms are also determined. It is shown that r must be expressible in the form P_n, where the x_i are rational numbers subject to conditions too lengthy to state here. For the special case $n=2$ the conditions are reduced to a simpler form.

W. H. Gage.

Klee, V. L., Jr. On a conjecture of Carmichael. Bull. Amer. Math. Soc. **53**, 1183–1186 (1947).

Let X denote the set of all x for which $\varphi(y)=\varphi(x)$ implies $y=x$. Let $\bar{x}=\prod_A p_i^{a_i}$ be in X, A denoting the set of indices i. If B and C denote disjoint subsets of A and $m=\prod_B p_i^{a_i-1}(p_i-1)\prod_C p_i^{c_i}$, $c_i\leqq a_i-1$ for i in C, then the author proves that, if p is a prime and $p-1=m$, we have $p|\bar{x}$. If moreover for every prime q that is a factor of p_j-1, j in B, we have $q|\bar{x}$, then $p^2|\bar{x}$. Carmichael's original conditions are special cases of this theorem; he proved that $\bar{x}>10^{37}$. The author uses this theorem to determine necessary factors of $\bar{x}$. These are first $2^2, 3^2, 7^2, 43^2, \cdots$. If now, e.g., 3^3 does not divide $\bar{x}$, every prime $6k+1$, where $(k, 6)=1$ or 2 and $k|\bar{x}$, is a necessary factor of $\bar{x}$. Hence x has the

factors 79, 157, 547, 1093, etc. From his three lists of necessary prime factors less than 10^7 the author infers $x > 10^{600}$.
N. G. W. H. Beeger (Amsterdam).

✱Gloden, A. Table des Solutions de la Congruence $x^4+1\equiv 0 \pmod{p}$ pour $500\,000 < p < 600\,000$. A. Gloden, Luxembourg, 1947. 12 pp.

This table is an extension, to the range of p indicated in the title, of a previous table [Mathematica, Timişoara **21**, 45–65 (1945); these Rev. **7**, 145]. *D. H. Lehmer.*

Mordell, L. J. On some Diophantine equations $y^2=x^3+k$ with no rational solutions. Arch. Math. Naturvid. **49**, no. 6, 143–150 (1947).

The equation of the title (k being an integer) has been much discussed with reference to its integral solutions. Much less is known about the possible rational solutions of such an equation. Fueter has, among other things, given certain classes of such equations with no rational solutions [Comment. Math. Helv. **2**, 69–89 (1930)]. The author proves the following result. The equation of the title has no rational solutions if k is a square-free positive integer and (1) $k\equiv 6, 15 \pmod{36}$; (2) the class number of the field $R(\sqrt{k})$ is not divisible by 3; (3) if T, U is the fundamental solution of the equation $t^2-ku^2=1$, so that U is the least positive value of u, then $U\not\equiv 0, \pm 1 \pmod 9$; (4) the class number h of the field $R(\sqrt{(-k/3)})$ is not divisible by 3; (5) the integer solutions of $p^2+(k/3)q^2=3^{2h}$, when $h\equiv 1 \pmod 3$, do not satisfy $q\equiv \pm 1 \pmod 9$; when $h\equiv -1 \pmod 3$, do not satisfy $q\equiv \pm 2(k/3)^2 \pmod 9$. It is noted that the values $k=6, 42, 51$ satisfy all of these conditions. The proof is quite straightforward and makes use only of the theory of ideals in quadratic fields.
H. W. Brinkmann (Swarthmore, Pa.).

Skolem, Th. Solutions of the equation $axy+bx+cy+d=0$ in algebraic integers. Avh. Norske Vid. Akad. Oslo. I. **1946**, no. 3, 8 pp. (1947).

Let K be a field of finite degree over the rationals, K_n a field of degree n over K, R and R_n the rings of integers in K and K_n. Consider the equation (*) $axy+bx+cy+d=0$ with coefficients in R, $(a, b, c)=1$, $a\neq 0$, $D=ad-bc\neq 0$. The principal results proved are: (*) is solvable in the residue class ring of any ideal in R; if R has a finite number of units, the number of solutions of (*) in R is finite, possibly zero; if R has an infinite number of units, the number of solutions of (*) in R is infinite or zero; for given R there exist n and R_n such that (*) has a solution in R_n (the proof gives an upper bound for the least value of n); if $(a, c)=1$, (*) is solvable in R if and only if D has a divisor congruent to $c \pmod a$; if $(a, c)=1$, there exists a ring R_n in which (*) is solvable if and only if D^n has a divisor $D_1\equiv c^n \pmod a$ in R. *I. Niven* (Eugene, Ore.).

Boomstra, W. Quadrangles whose sides and diagonals can be represented by integers. Nieuw Tijdschr. Wiskunde **35**, 117–120 (1947). (Dutch)

Faddeev, D. K. On the characteristic equations of rational symmetric matrices. Doklady Akad. Nauk SSSR (N.S.) **58**, 753–754 (1947). (Russian)

The result of the paper has its origin from the following problem. Find a necessary and sufficient condition for an equation which can be the characteristic equation of a rational symmetric matrix. Let α be an algebraic number which is a characteristic root of a rational symmetric matrix A. Then $f(\alpha)$, where $f(x)$ is a polynomial with rational coefficients, is a characteristic root of the rational symmetric matrix $f(A)$. Therefore every element of the field $R(\alpha)$, the extension of the rational field R by adjunction of α, is a characteristic root of a rational symmetric matrix. The theorem proved by the author is the following. Let $k=R(\alpha)$ be a totally real field of degree $n\leqq 7$. Suppose that there exists a number λ such that the ideal (λ) is a square of an ideal in k and the conjugates of λ take any prescribed signs. Then every element of k is a characteristic root of a certain symmetric matrix of order n. We can establish a correspondence such that an integer of the field is mapped into a symmetric matrix with integer elements.
L.-K. Hua (Princeton, N. J.).

Rados, Gustav. Über die Elementarteiler der adjungirten Formen einer bilinearen Form mit ganzen Coefficienten. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 333–351 (1941). (Hungarian. German summary)

Suppose that $B=\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_iy_j$ is a bilinear form whose coefficients are rational integers and that $1\leqq k\leqq n$. Any two combinations $\lambda=(i_1, \cdots, i_k)$ and $\mu=(j_1, \cdots, j_k)$ of the integers $(1, \cdots, n)$ taken k at a time determine a minor $B_{\lambda\mu}^{(k)}$ of B; the adjoint forms of B are defined by $\mathrm{Adj}^{(k)}(B)=\sum_\lambda\sum_\mu B_{\lambda\mu}^{(k)}x_\lambda y_\mu$. If $(e_1, \cdots, e_n)$ is any set of numbers and $\mu=(i_1, \cdots, i_k)$ is a combination of the integers $(1, \cdots, n)$ taken k at a time, a combinatorial product $E_\mu^{(k)}$ is defined by $E_\mu^{(k)}=\prod_{j=1}^{k} e_{i_j}$. The author's principal result is the assertion that if the determinant of B does not vanish, and if the elementary divisors of B are $e_1, \cdots, e_n$, then there exist two unimodular forms P and Q such that

$$\mathrm{Adj}^{(k)}(PBQ)=\mathrm{Adj}^{(k)}(P)\ \mathrm{Adj}^{(k)}(B)\ \mathrm{Adj}^{(k)}(Q)=\textstyle\sum_\mu E_\mu^{(k)}x_\mu y_\mu.$$

The elementary divisors of $\mathrm{Adj}^{(k)}(B)$ are then easily calculable from the combinatorial products $E_\mu^{(k)}$. Certain special cases and certain numerical examples are also studied. With minor modifications the results extend to the case in which the determinant of B vanishes. *P. R. Halmos.*

Mahler, K. On the adjoint of a reduced positive definite ternary quadratic form. Acad. Sinica Science Record **2**, 21–31 (1947).

Let $a_1x_1^2+a_2x_2^2+a_3x_3^2+2b_1x_2x_3+2b_2x_3x_1+2b_3x_1x_2$ be a positive definite form reduced in the sense of Seeber and Minkowski. Let D be the discriminant of the form and $A_1, A_2, A_3, B_1, B_2, B_3$ be the coefficients of the adjoint form. The author proves the best possible inequality $A_1A_2A_3\leqq \frac{9}{4}D^2$ and states several others such as $D\leqq a_1a_2a_3-4b_1b_2b_3\leqq 2D$, $\frac{3}{4}a_2a_3\leqq A_1\leqq a_2a_3$, $A_2\geqq \frac{3}{4}A_3$, $|B_1|\leqq \frac{1}{2}A_2$. *H. S. A. Potter.*

Rogers, C. A. Existence theorems in the geometry of numbers. Ann. of Math. (2) **48**, 994–1002 (1947).

Let S be a closed bounded n-dimensional star body ($n\geqq 2$), symmetrical about the origin O, with a volume $V(S)$. The author gives a simple proof of a theorem of C. L. Siegel (except for an ϵ term) [same Ann. (2) **46**, 340–347 (1945); these Rev. **6**, 257]. This theorem is used (1) to deduce a Minkowski's assertion: if $V(S)<2\zeta(n)$ (Riemann zeta function), it is possible to find a lattice $\mathfrak{G}$ of unit determinant such that O is the only point of $\mathfrak{G}$ in S; (2) to prove a new theorem: if $V(S)<2n\zeta(n)l^{-1}(1-l^{-n})^{-1}=\sigma(n)$, it is possible to find a lattice $\mathfrak{G}$ of unit determinant such that $\lambda_1\cdots\lambda_n>1$, where λ_i is the least value of λ such that the body λS contains at least i linearly independent points of $\mathfrak{G}$. This theorem is used to establish the following result: there exists a positive definite quadratic form $Q(u)$ of unit determinant

in n variables u_i such that $Q(u)\geqq\pi^{-1}\{\sigma(n)\Gamma(1+n/2)\}^{2/n}$ for all the lattice points $u\neq O$. See also Davenport and Rogers [Duke Math. J. 14, 367–375 (1947); these Rev. 9, 11].

V. Knichal (Prague).

Davenport, H. The geometry of numbers. Math. Gaz. 31, 206–210 (1947).

This is the report of an address, the purpose of which was to introduce the theory of the geometry of numbers to non-specialists. Minkowski's proof of the fundamental lattice point theorem for convex bodies is given. This is followed by applications among which is the Hermite proof of the Lagrange result that every integer can be represented as the sum of at most four integral squares. *D. Derry.*

Davenport, H. A historical note. J. London Math. Soc. 22, 100–101 (1947).

In 1935 the author proved the following theorem [same J. 10, 30–32]. Let p be a prime, let $\alpha_1, \cdots, \alpha_m$ be m different residue classes mod p and let $\beta_1, \cdots, \beta_n$ be n different residue classes mod p. Let $\gamma_1, \cdots, \gamma_l$ be all residues which are representable in the form $\alpha_i+\beta_j$, $i=1, \cdots, m$; $j=1, \cdots, n$. Then $l\geqq\min(m+n-1, p)$. He now finds that this theorem was proved by Cauchy in 1813 [J. École Polytech. cahier 16, tome 9, 99–123 = Oeuvres, ser. 2, v. 1, pp. 39–63].

P. Erdös (Syracuse, N. Y.).

Shepherdson, J. C. On the addition of elements of a sequence. J. London Math. Soc. 22, 85–88 (1947).

The author proves the following theorem. Let $\alpha_1, \cdots, \alpha_m$ be elements of an additive Abelian group G. Let $n\leqq m$ and denote by $\gamma_1, \cdots, \gamma_l$ those elements of G which can be written in the form α_i or $\alpha_i+\alpha_j$, $1\leqq i\leqq m$; $1\leqq j\leqq n$. Then either $l\geqq m+n$ or there exists a k such that all elements of the cyclic group generated by α_k belong to the γ's. This generalizes the result quoted in the preceding review. The author also deduces two corollaries. *P. Erdös.*

Gupta, Hansraj. An asymptotic formula in partitions. J. Indian Math. Soc. (N.S.) 10, 73–76 (1946).

Let $p(n, m)$ denote the number of partitions of n into exactly m parts and let $p(n)$ denote the number of unrestricted partitions of n. The author gives a proof of the following theorem. Let $t>0$ and let n and m tend to infinity in such a way that $\lim_{n,m\to\infty}(6n)^{\frac12}\exp(-\pi m(6n)^{-\frac12})=t$; then $(6n)^{\frac12}p(n, m)/p(n)\sim te^{-t/\pi}$. This is equivalent to a previous result of Auluck, S. Chowla and the author [same J. (N.S.) 6, 105–112 (1942); these Rev. 4, 211]. *D. H. Lehmer.*

Dyson, F. J. On simultaneous Diophantine approximations. Proc. London Math. Soc. (2) 49, 409–420 (1947).

This paper is concerned with generalizations of Khintchine's "Übertragungsprinzip" [see Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, chap. 5]. Let θ_{ij} ($1\leqq i\leqq n$, $1\leqq j\leqq m$) be real numbers. Let

$$f_i=\theta_{i1}x_1+\cdots+\theta_{im}x_m-X_i, \qquad 1\leqq i\leqq n.$$

The matrix θ_{ij} is said to admit the (nonnegative) exponents $\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_m$ if, for all $A>0$, $B>0$, the inequalities

$$|Af_i|^{\beta_j}|Bx_j|^{\alpha_i}\leqq 1, \quad 1\leqq i\leqq n,\ 1\leqq j\leqq m,$$

are simultaneously soluble in integers $x_1, \cdots, x_m, X_1, \cdots, X_n$ with not all the x's zero. Let ζ be defined by

$$\zeta+\sum_{\alpha_i>\zeta}(\alpha_i-\zeta)=\sum_{\beta_j>-\zeta}(\beta_j+\zeta).$$

Let $\rho_i=\max(0, \alpha_i-\zeta)$ for $1\leqq i\leqq n$; $\sigma_j=\max(0, \beta_j+\zeta)$ for $1\leqq j\leqq m$. The main theorem is that if the matrix θ admits the exponents (α_i, β_j) then its transpose θ' admits the exponents (σ_j, ρ_i). This includes Khintchine's original theorem and various extensions of it. The proof is based on the "Schubfachschluss." [An alternative method would be to follow Mahler's very simple proof of Khintchine's original theorem, Rec. Math. [Mat. Sbornik] N.S. 1(43), 961–962 (1936).] *H. Davenport* (Stanford University, Calif.).

Rédei, L. Über eine diophantische Approximation im bereich der algebraischen Zahlen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 61, 460–470 (1942). (Hungarian. German summary)

The author proves the following theorem. Let α be a real algebraic number, $|\alpha|=1$. A necessary and sufficient condition that the sequence $\alpha^n-[\alpha^n]$ should converge is that α is an algebraic integer and all conjugates of α are less than 1 in absolute value. If this condition is satisfied, then $\lim_{n\to\infty}(\alpha^n-[\alpha^n])=0$. [The same result was proved by Vijayaraghavan, Proc. Cambridge Philos. Soc. 37, 349–357 (1941); these Rev. 3, 274.] *P. Erdös* (Syracuse, N. Y.).

Rédei, L. Zwei Lückensätze über Polynome in endlichen Primkörpern mit Anwendung auf die endlichen Abelschen Gruppen und die Gaussischen Summen. Acta Math. 79, 273–290 (1947).

Let P denote the $GF(p)$, $p>2$; by a P-polynomial is meant a polynomial with coefficients in P. The following theorems are proved. (1) A P-polynomial

$$f(x)=x^{p-1}+\gamma x^{\frac12(p-1)}+\cdots$$

is a product of linear P-polynomials if and only if

$$f(x)=x^{\frac12(p-1)t}(x^{\frac12(p-1)}-1)^u(x^{\frac12(p-1)}+1)^v,$$

where $t+u+v=2$. (2) A P-polynomial

$$g(x)=x^{\frac12(p-1)}+\gamma x^n+\cdots\neq x^{\frac12(p-1)}\pm1$$

($n\leqq\frac14(p-1)$, $\gamma\neq0$, $g(0)\neq0$) is a product of linear P-polynomials if and only if $4|p-1$ and

$$g(x)=(x^{\frac14(p-1)}-1)^t(x^{\frac14(p-1)}-1)^u(x^{\frac14(p-1)}-\sigma)^v(x^{\frac14(p-1)}+\sigma)^w,$$

where $\sigma^2=-1$, $t+u=v+w=1$.

Of the applications we quote the following. (4) If a noncyclic Abelian group G can be expressed in the form HK, where H and K are complexes each of which contains p elements (including the identity), then either H or K is a group. (5) Let ρ denote a primitive pth root of unity and put $A=\rho^{x_1}+\cdots+\rho^{x_p}$, where the x_i are arbitrary integers. Then (excluding the trivial cases $A=0$, $A=p$) A is divisible by $(1-\rho)^{\frac12(p-1)}$ only if $A=\pm\Gamma$ or $\frac12(p\pm\Gamma)$, where $\Gamma=\sum_{k=1}^{p}\rho^{k^2}$.

L. Carlitz (Durham, N. C.).

Selberg, Atle. On an elementary method in the theory of primes. Norske Vid. Selsk. Forh., Trondhjem 19, no. 18, 64–67 (1947).

Let $p_1, p_2, \cdots$ be an arbitrary set of primes. Denote by $N(n)$ the number of integers $m\leqq n$ which are not divisible by any of the p's. The author obtains (and in fact improves) in a simple way the estimate for the upper bound of $N(n)$ which was previously obtained by Brun's method. His method seems to have great possibilities. *P. Erdös.*

Selberg, Atle. Contributions to the theory of Dirichlet's L-functions. Skr. Norske Vid. Akad. Oslo. I. 1946, no. 3, 62 pp. (1946).

The author gives further examples of the well-known analogy between the behaviour of $\zeta(s)$ as a function of t and

the behaviour of the Dirichlet L-series $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ ($\chi(n)$ a primitive character mod k) as a function of k. To avoid tedious arithmetical complications, he limits himself to the case k prime. His first result (theorems 1 and 2) is an asymptotic formula for the sum $\sum_\chi L(s,\chi)L(s',\bar\chi)\chi(\mu_1)\bar\chi(\pm\mu_2)$, where χ runs through all primitive characters mod k, s and s' lie inside the critical strip and μ_1 and μ_2 are positive integers prime to each other.

He next proves theorem 3. Let $|t_1|\leqq K^{\frac{1}{2}-\epsilon}$ and

$$(\log k)^{-1}<H<(\log k)^{-\frac{1}{2}};$$

then there is an $A=A(\epsilon)>0$ and a $k_0=k_0(\epsilon)$ such that for $k>k_0$ at least $k(1-A(H\log k)^{-\frac{1}{2}})$ of the functions $L(s,\chi)$ have zeros on the line $\sigma=\frac{1}{2}$ in the interval $t_1<t<t_1+H$. This implies that for $|t|\leqq k^{\frac{1}{2}-\epsilon}$ more than "γ percent" of the zeros of $\prod_\chi L(s,\chi)$ lie on the line $\sigma=\frac{1}{2}$, γ being an absolute constant.

The next step is to prove theorem 4. Let $0<\epsilon<\frac{1}{4}$, $-k^{\frac{1}{2}-\epsilon}<t_1<t_2<k^{\frac{1}{2}-\epsilon}$, $t_2-t_1\geqq\log^{-1}k$, $\sigma\geqq\frac{1}{2}+\log^{-1}k$, and let $N_\chi(\sigma;t_1,t_2)$ be the number of zeros $\beta+\gamma i$ of $L(s,\chi)$ in the rectangle $\sigma\leqq\beta<1$, $t_1\leqq t\leqq t_2$. Then

$$\sum_\chi N_\chi(\sigma;t_1,t_2)=O(k^{1-\frac{1}{2}(\sigma-\frac{1}{2})\epsilon}(t_2-t_1)\log k).$$

Expressed roughly this means that "almost all" zeros of the functions $L(s,\chi)$ are near the line $\sigma=\frac{1}{2}$.

Finally the author gives inequalities for $S(t,\chi)$, the principal value of the argument of $L(s,\chi)$ at the point $s=\frac{1}{2}+ti$. The function $S(t,\chi)$ is important because it gives the dominant error term in von Mangoldt's formula for the number of zeros of $L(s,\chi)$ in the critical strip. The most striking results are as follows. If $L(s,\chi)\neq 0$ for $\sigma>\frac{1}{2}$, then

$$S(t,\chi)=O\left(\frac{\log(k(1+|t|))}{\log\log(k(3+|t|))}\right).$$

If $|t|\leqq k^{\frac{1}{2}-\epsilon}$, then $\sum_\chi S(t,\chi)=O(k)$. If $|t|<k^{\frac{1}{2}-\epsilon}$ and if r is a fixed positive integer, then

$$\sum_\chi |S(t,\chi)|^{2r}=(2r)!(r!)^{-1}(2\pi)^{-r}k((\log\log k)^r+O(\log\log k)^{r-\frac{1}{2}}).$$

H. Heilbronn (Bristol).

Chowla, S. On an unsuspected real zero of Epstein's zeta function. Proc. Nat. Inst. Sci. India 13, no. 4, 1 p. (1947).

Let the Epstein ζ-function be defined by

$$\zeta(s,Q)=\sum_{x,y=-\infty}^{\infty}{}' Q(x,y)^{-s},$$

where $Q(x,y)=ax^2+bxy+cy^2$ is a positive definite form with integral coefficients and the fundamental discriminant b^2-4ac; here the prime on the summation indicates that $(x,y)\neq(0,0)$. Davenport and Heilbronn [J. London Math. Soc. 11, 181–185 (1936)] showed that this ζ-function has an infinity of zeros with $\Re(s)>1$. The author considers the special case $Q(x,y)=x^2+dy^2$ for arbitrary d and indicates a proof of the result that there exists a d_0 such that corresponding to each $d>d_0$ there is a real zero $s(d)$ of this ζ-function such that $s(d)\sim 1-3/(\pi d^{\frac{1}{2}})$ as $d\to\infty$. Thus for large d, $\frac{1}{2}<s(d)<1$, so that the analogue of the Riemann hypothesis is false. The proof is said to be similar to that used previously by the author [Quart. J. Math., Oxford Ser. 5, 302–303 (1934)]. However, in that paper the author was able to show that $s(d)>1-c/\log d$, for some c, by assuming the existence of infinitely many imaginary quadratic fields with the class number one; it is not clear how he will establish this inequality in the present case. For $d=5$, Potter and Titchmarsh [Proc. London Math. Soc. (2) 39, 372–384 (1935)] had already suspected that the seventh and eighth zeros were off the line $\Re(s)=\frac{1}{2}$.

L. Schoenfeld (Cambridge, Mass.).

Chowla, S. On a problem of analytic number theory. Proc. Nat. Inst. Sci. India 13, 231–232 (1947).

It is a well-known consequence of Dirichlet's class-number formula that, if p is a prime of the form $4n+3$, there are more quadratic residues than nonresidues between 0 and $\frac{1}{2}p$. No elementary proof of this has yet been given. The note contains a proof which is certainly very simple, but it cannot be considered as elementary, in the sense appropriate to the problem, since it uses infinite series. [A similar proof was given by K.-L. Chung, Bull. Amer. Math. Soc. 47, 514–516 (1941); these Rev. 3, 66.] *H. Davenport.*

Chowla, S. A theorem in analytic number theory. Proc. Nat. Inst. Sci. India 13, 97–99 (1947).

The author announces the following theorem. Let k and r be given positive integers. Then almost all positive integers n are of the form $n=\prod_{i=1}^{r}p_i^{2a_i+1}$, where each prime p is of the form $kx-1$. The proof is given for $r=2$ and depends on the prime number theorem. For $r>2$ the proof is said to depend on a recent result of S. Selberg [Skr. Norske Vid. Akad. Oslo. I. 1942, no. 5; these Rev. 6, 57]. It follows from the general theorem that, for almost all n, Ramanujan's function $\tau(n)$ is divisible by an arbitrarily high power of p, for $p=2, 3, 5, 7, 23$ and 691. *D. H. Lehmer.*

Chowla, S. Note on a certain arithmetical sum. Proc. Nat. Inst. Sci. India 13, no. 5, 1 p. (1947).

Let $\sigma(n)$ denote the sum of the divisors of n and define the r-fold sum $S_r(n)$ by

$$S_r(n)=\sum\sigma(u_1)\sigma(u_2)\cdots\sigma(u_r),$$

where the sum extends over all positive integers u such that $u_1+u_2+u_3+\cdots+u_r=n$. When n is a prime p the author finds that

$$\begin{aligned}(1)\qquad 12S_2(p)&=(p-1)(p+1)(5p-6),\\ 192S_3(p)&=(p-1)^2(p+1)(p-2)(7p-9),\end{aligned}$$

and states that $S_4(p)$ and $S_5(p)$ are also polynomials (of degrees 7 and 9). The result (1) is due to Ramanujan.

D. H. Lehmer (Berkeley, Calif.).

Bambah, R. P., and Chowla, S. A congruence property of Ramanujan's function $\tau(n)$. Proc. Nat. Inst. Sci. India 12, 431–432 (1946).

Another proof is given of the fact that $\tau(n)\equiv\sigma(n)$ (mod 3) when n is not divisible by 3 [for other proofs cf. Gupta, J. Indian Math. Soc. (N.S.) 9, 59–60 (1945); these Rev. 8, 10]. This proof is based on the fact that the number of representations of an odd integer n as the sum of 24 squares is a linear combination of $\tau(n)$ and $\sigma_{11}(n)$.

D. H. Lehmer (Berkeley, Calif.).

Bambah, R. P., and Chowla, S. On a function of Ramanujan. Proc. Nat. Inst. Sci. India 12, no. 8, 1 p. (1946).

The authors announce the following congruences for Ramanujan's function $\tau(n)$:

$$\begin{aligned}\tau(n)&\equiv\sigma_{11}(n)\pmod{256}\qquad(n\text{ odd}),\\ \tau(p)&\equiv -4p\sigma_9(p)+5p^2\sigma_7(p)\pmod{125},\end{aligned}$$

where p is a prime not equal to 5. *D. H. Lehmer.*

Bambah, R. P., and Chowla, S. On numbers which can be expressed as a sum of two squares. Proc. Nat. Inst. Sci. India 13, 101–103 (1947).

By a straightforward elementary construction the authors prove the existence of a constant C such that for $x>0$ there is at least one integer between x and $x+Cx^{\frac{1}{4}}$ which can be expressed as a sum of two squares. They remark that this is obviously very far from the probable truth.

P. T. Bateman (New Haven, Conn.).

Chowla, S. Modular equations as solutions of algebraic differential equations of the sixth order. Proc. Nat. Inst. Sci. India 13, 169–170 (1947).

It is shown that modular equations are solutions of algebraic differential equations. *H. S. Zuckerman.*

✶Siegel, Carl Ludwig. Indefinite quadratische Formen und Modulfunktionen. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 395–406. Interscience Publishers, Inc., New York, 1948. $5.50.

Let $\mathfrak{S}$ be the matrix of an even quadratic form $\mathfrak{S}[\mathfrak{x}]$ of signature (n, ν) in $m=n+\nu$ variables x_i and with determinant $|\mathfrak{S}|=S>0$ so that ν is even. Then $\mathfrak{S}[\mathfrak{x}]=r-\rho$, where $r=t_1^2+\cdots+t_n^2$, $\rho=\tau_1^2+\cdots+\tau_\nu^2$ and the t_i and τ_i are homogeneous linear functions in the x_i with real coefficients. If z and ζ are complex variables in the upper and lower half-plane, respectively, and $\mathfrak{H}[\mathfrak{x}]=r+\rho$ then the quadratic form $\mathfrak{R}[\mathfrak{x}]=zr-\zeta\rho$ has the matrix $\mathfrak{R}=\frac{1}{2}(z+\zeta)\mathfrak{S}+\frac{1}{2}(z-\zeta)\mathfrak{H}$ with a positive imaginary part. The author first proves by well-known methods that the function $f(z,\zeta)=\sum \exp \pi i\mathfrak{R}[\mathfrak{G}]$, where $\mathfrak{G}$ ranges over all integral vectors, satisfies

$$f(\hat{z}, \hat{\zeta})=\omega(cz+d)^{\frac{1}{2}n}(c\zeta+d)^{\frac{1}{2}\nu}f(z,\zeta)$$

if z and ζ are transformed by the simultaneous modular substitutions

$$z\rightarrow\hat{z}=(az+b)/(cz+d),\quad \zeta\rightarrow\hat{\zeta}=(a\zeta+b)/(c\zeta+d),$$

and $c\mathfrak{S}^{-1}$ is even. Here $\omega^4=1$.

The author then shows how a modular form $F(z)$ in one variable z can be attached to the form $\mathfrak{S}[\mathfrak{x}]$ if $\mathfrak{S}$ is indefinite (so that $n\nu>0$) (the two special cases (a) $m<4$, $\mathfrak{S}[\mathfrak{x}]$ is a zero form; (b) $m=4$, $\mathfrak{S}[\mathfrak{x}]$ is a zero form, S is a square, have to be excluded). This is achieved in the following way. All positive solutions of $\mathfrak{H}\mathfrak{S}^{-1}\mathfrak{H}=\mathfrak{S}$ constitute an $n\nu$-dimensional space H. If $\mathfrak{F}$ is a real automorph of $\mathfrak{S}[\mathfrak{x}]$ then H is mapped onto itself by the mapping $\mathfrak{H}\rightarrow\mathfrak{H}[\mathfrak{F}]$. The quadratic differential form $ds^2=\frac{1}{8}$ trace $(\mathfrak{H}^{-1}d\mathfrak{H}\ \mathfrak{H}^{-1}d\mathfrak{H})$ is invariant under the group Ω of all real automorphs $\mathfrak{F}$ and defines a Riemann metric in H. The subgroup $\Gamma(\mathfrak{S})$ of Ω consisting of all integral automorphs of $\mathfrak{S}$ has a fundamental domain $H(\mathfrak{S})$ in H. It has been proved by the author in an earlier paper [Abh. Math. Sem. Hansischen Univ. 13, 209–239 (1940); these Rev. 2, 148] that the volume V of $H(\mathfrak{S})$ is finite (unless $m=2$ and S is a square) and that the integral $J(f)=\int f dv$ over $H(\mathfrak{S})$ converges. The author now proves that

$$F(z)=\Gamma(\tfrac{1}{2}n)\Gamma^{-1}(\tfrac{1}{2}m)V^{-1}(z-\zeta)^{1-1/n}(\partial/\partial x)^{\frac{1}{2}\nu}\{(z-\zeta)^{\frac{1}{2}m-1}J(f)\}$$

is a function of z only and is a modular form satisfying

$$F(z+1)=F(z),\quad F((az+b)/(cz+d))=\omega(cz+d)^{\frac{1}{2}m}F(z),$$

if $c\mathfrak{S}^{-1}$ is even and $c>0$. *H. D. Kloosterman* (Leiden).

Pollaczek, Félix. Relations entre les dérivées logarithmiques de Kummer et les logarithmes π-adiques. Bull. Sci. Math. (2) 70, 199–218 (1946).

Let ζ be a primitive pth root of unity, p an odd prime, and let $\alpha=\alpha(\zeta)=c_0+c_1(1-\zeta)+\cdots+c_m(1-\zeta)^m$ be an element of the cyclotomic field $R(\zeta)$, where $c_0,\cdots,c_m$ are rational numbers with denominators prime to p. For $\alpha(1)=c_0\equiv1 \pmod p$, Kummer defined the logarithmic derivatives $l_\nu(\alpha)$, $\nu=1,\cdots,p-1$, which are independent of the particular polynomial representation of α:

$$l_\nu(\alpha)\equiv\left[\frac{d^\nu\log\alpha(e^v)}{dv^\nu}\right]_{v=0}+\delta_{\nu,p-1}\frac{c_0-1}{p}$$

$$=[\log\alpha(e^v)]_0^{(\nu)}+\delta_{\nu,p-1}\frac{c_0-1}{p}\pmod p,$$

where δ is the Kronecker delta. He employed them in deriving congruences related to Fermat's last theorem and derived the following expression for the inversion factor for $p$ic residue symbols, for certain cases of $R(\zeta)$:

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)^{-1}=\zeta^K,\quad K\equiv\sum_{\nu=1}^{p-1}(-1)^\nu l_\nu(\alpha)l_{p-\nu}(\beta)\pmod p.$$

This was proved in other cases by Takagi. Hasse obtained expressions for $K \pmod p$ in terms of π-adic logarithms:

$$\log_\pi(1+\pi a)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}(1-\zeta)^na^n,\quad \pi=1-\zeta,$$

where a is an algebraic number with denominator prime to p, so that the series is π-adically convergent. His proofs employed (1) the theory of the cyclotomic fields and Kummer extensions of them and (2) expressions for inversion factors in terms of norm-residue symbols. Without employing either (1) or (2), the author shows the equivalence of the Kummer-Takagi expression for K and Hasse's. He accomplishes this by using the p-adic limits

$$L_\nu(\alpha)=\lim_{n\to\infty}[\log\alpha(e^v)]_0^{\nu p^n},$$
$$\nu\equiv0,1,\cdots,p-2 \pmod{p-1},$$

whose existence and other properties he proves. In particular, the equivalence mentioned follows from expressions for the L_ν in terms of π-adic logarithms and Lagrange resolvents in the π-adic field $R_\pi=R_p(\zeta)$. The author also derives p-adic equations which are generalizations of congruences due to Kummer, related to Fermat's last theorem.

R. Hull (Lafayette, Ind.).

ANALYSIS

Good, I. J. A note on positive determinants. J. London Math. Soc. 22, 92–95 (1947).

Let Δ be the determinant with entries $\phi_j(x_k)$ where the functions ϕ_j are integrable in the interval $[A, B]$; choose subintervals $I_1,\cdots,I_n$ of $[A, B]$ such that the end points of I_r are to the left of the corresponding end points of I_{r-1} while I_r and I_{r-2} are disjoint. Let D be the determinant whose entries are $\int_{I_k}\phi_j(u)du$. The principal result is that, if $\Delta\geqq0$ whenever $B\geqq x_1\geqq x_2\geqq\cdots\geqq x_n\geqq A$, then $D\geqq0$. Several applications of this are given. If f is nonnegative,

if $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$, and if the intervals I_r obey the conditions above, then the determinant with entries $\int_{I_k} f(x)x^{\lambda_i}dx$ is nonnegative. If $F(t, x)/F(u, x)$ is increasing in x when $t \geqq u$, then $\int_a^b F(t, x)dx / \int_c^d F(t, x)dx$ is increasing in t if d and a lie between b and c. *R. C. Buck* (Providence, R. I.).

Salem, R. Sur une extension du théorème de convexité de M. Marcel Riesz. Colloquium Math. 1, 6–8 (1947).

The author gives the shortest proof yet found for Thorin's generalization [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 8, no. 14 (1939)] of Riesz's convexity theorem and of a generalization in which the absolute value of an analytic function is replaced by a subharmonic function. The theorem now reads: let $f(z) = f(z_1, \cdots, z_n)$ be subharmonic; V a bounded domain in Euclidean n-space with coordinates $v = (v_1, \cdots, v_n)$; $M(\alpha_1, \cdots, \alpha_n)$ the maximum of $f(z)$ for $|z_k| = v_k^{\alpha_k}$, $k = 1, \cdots, n$ and $v \epsilon V$; if $0 < A \leqq v_k \leqq B < \infty$ for $k = 1, \cdots, n$, then $\log M(\alpha)$ is a convex function of α in $-\infty < \alpha_k < \infty$ $(k = 1, \cdots, n)$. Proof. Put $z_k = e^{\alpha_k \xi_k + i\eta_k}$, with ξ in a bounded domain D corresponding to $(e^{\xi_1}, \cdots, e^{\xi_n})$ in V. Put $\alpha_k = a_k + \lambda_k \log t$, $t > 0$; $M(\alpha)$ becomes a function $M(t)$ of t and it is to be shown that $\log M(t)$ is a convex function of $\log t$. Put

$$I_p(t) = \int |f(e^{\xi_1(a_1+\lambda_1 \log t)+i\eta_1}, \cdots, e^{\xi_n(a_n+\lambda_n \log t)+i\eta_n})|^p d\xi d\eta,$$

with $p \geqq 1$ and the integral over $0 \leqq \eta_k \leqq 2\pi$, $\xi \epsilon D$. It is easy to show that if t is now a complex variable, $I_p(t)$ is a uniform subharmonic function of t and $I_p(t) = I_p(|t|)$. Hence $\log I_p(|t|)$ is convex in $\log |t|$ and since $\{I_p(t)\}^{1/p} \to M(t)$ as $p \to \infty$, the result follows. *R. P. Boas, Jr.*

Calculus

Hoheisel, Guido. Funktionalgleichung und Differenzierbarkeit bei den trigonometrischen Funktionen. Math. Ann. 120, 10–11 (1947).

The author shows how the limit $\lim_{x\to 0} x^{-1} \sin x$ can be found without using concepts like length or area depending on the idea of integration. He uses only the convergence of monotonic bounded sequences, the functional equation for $\sin x$, and elementary facts about the equation $f(x+y) = f(x) + f(y)$. *R. P. Boas, Jr.* (Providence, R. I.).

Ghosh, P. K. On $(C, 1)$-convergent integrals and their application to mathematical physics. Bull. Calcutta Math. Soc. 39, 19–29 (1947).

The author extends familiar theorems of the calculus of definite integrals to integrals that are Cesàro summable, and gives applications to quantum-mechanical problems (particularly scattering problems). *R. P. Boas, Jr.*

Pars, L. A. A note on the envelope of a certain family of curves. J. London Math. Soc. 22, 25–31 (1947).

For the two-parameter family of curves $y/c = f((x-a)/c)$, assume $f''(x)$ exists for all x, $f''(x) \geqq m > 0$, m constant, $f(0) = 0$, $f'(0) > 0$. It is proved by elementary methods that the one-parameter subfamily passing through any fixed point (x_0, y_0) in the upper half-plane has an envelope $(x \neq x_0)$ such that two members of the subfamily pass through any given point above the envelope, and none through any point below the envelope. Also, that the tangent at the point of contact and the tangent at (x_0, y_0) meet on the y-axis. The latter result was known for the family of catenaries $y/c = \cosh [(x-a)/c]$, and the former for the same family and for the family of parabolas $y/c = 1 + \frac{1}{4}(x-a)^2/c^2$. Both families are solutions of problems in the calculus of variations. *A. B. Brown* (Flushing, N. Y.).

Vegas Pérez, Angel. Short deduction of Stirling's formula for the calculation of $n!$. Revista Acad. Ci. Madrid 36, 126–129 (1942). (Spanish)

The author uses the Euler-Maclaurin summation formula. *R. P. Boas, Jr.* (Providence, R. I.).

Sharma, A. On the minimal interval of ξ in the second mean-value theorem. Proc. Benares Math. Soc. (N.S.) 7, no. 2, 33–40 (1945).

Extension au deuxième théorème de la moyenne de la méthode et du résultat de Tchakaloff. Applications et exemples. *J. Favard* (Paris).

Deaux, Roland. Sur le champ de moments. Simon Stevin 25, 172–178 (1947).

The author gives two methods of finding a vector S knowing the moment vectors of S with respect to three points of the space. *W. J. Nemerever* (Ann Arbor, Mich.).

Theory of Sets, Theory of Functions of Real Variables

Bagemihl, Frederick. On the partial products of infinite products of alephs. Amer. J. Math. 70, 207–211 (1948).

Tarski [Fund. Math. 7, 1–14 (1925)] has shown that the product $\mathfrak{p}$ of a strictly increasing sequence of type $\lambda = \omega^\delta$ of transfinite cardinals satisfies the equation $\mathfrak{p} = \mathfrak{p}^{\mathfrak{l}}$ (where $\mathfrak{l}$ is the cardinal of the set of ordinals less than λ). The author shows that this equation is still satisfied when the sequence is nondecreasing but that it need not be satisfied when λ is not of the form ω^δ. When the sequence is nondecreasing, $\mathfrak{p}$ is the product of the partial products for any λ but this need not be the case if the sequence is not monotone. [On p. 460 of the same vol. the author has indicated several mistakes in the printing of some formulae.] *J. Todd.*

Sierpinski, W. Sur certains systèmes déterminants. Actas Acad. Ci. Lima 10, 17–23 (1947).

Let S be a Suslin scheme possessing the following property P: (1) every set of S is a closed linear interval $l \cdot 2^{1-k} \leqq x \leqq (l+1) \cdot 2^{1-k}$, where l is an integer; (2) the scheme S is monotone; (3) for every natural number k there exist only a finite number of different sets of S with k indices. The author proves the following theorem. In order that a linear set M be the nucleus of a Suslin scheme with the property P, it is necessary and sufficient that M be a nonempty, bounded, analytic set. [The paper contains many misprints.] *A. Rosenthal* (Lafayette, Ind.).

Kestelman, H. The convergent sequences belonging to a set. J. London Math. Soc. 22, 130–136 (1947).

A null sequence $\{\lambda_n\}$ is said to belong to a set E if, for some c, $c + \lambda_n \epsilon E$ for all large n. The set E is said to be universal if every null sequence belongs to it. If E contains an open set, it is universal. If E is of the form $I - Z$, where Z is of measure zero, or of first category, and I is open, then E is universal. If E is universal, then the closure of E con-

tains an open set. An example is given of a decomposition of an interval into two disjoint universal sets. These notions are closely related to the set of "distances" of a set. If E has the property that every point of a sufficiently small sphere about the origin appears as the difference of two points of E, then one writes $E\epsilon\mathfrak{S}$. A theorem of Steinhaus [Fund. Math. 1, 93–104 (1920)] states that, if E is a closed set of positive measure, $E\epsilon\mathfrak{S}$. Using methods of a previous paper [Fund. Math. 34, 144–147 (1947); these Rev. 9, 188], the author proves several extensions of this. In particular, if E is a closed set of positive measure, then every null sequence which converges rapidly enough belongs to E; however, E need not be universal, although a universal set must belong to $\mathfrak{S}$. *R. C. Buck* (Providence, R. I.).

Dubrovskiĭ, V. M. On the basis of a family of completely additive functions of sets and on the properties of uniform additivity and equi-continuity. Doklady Akad. Nauk SSSR (N.S.) 58, 737–740 (1947). (Russian)

The author considers an abstract set A and a given family $\mathfrak{M}$ of subsets of A such that $\mathfrak{M}$ contains A and 0 and is closed under the formation of differences and of countable unions. He further considers a family $\{\Phi_\lambda(E)\}$, $\lambda\epsilon\Lambda$, of completely additive (real) functions defined on the family $\mathfrak{M}$. The following definitions are introduced. (1) The family $\{\Phi_\lambda(E)\}$ enjoys the property of uniform additivity if, for every family $\{E_k\}_{k=1}^{\infty}$ of disjoint sets in $\mathfrak{M}$, $\lim_{n\to\infty}\Phi_\lambda(E_{n+1}\cup E_{n+2}\cup\cdots)=0$, uniformly for all $\lambda\epsilon\Lambda$. (2) The completely additive nonnegative real-valued function M, defined on $\mathfrak{M}$, is a basis for the family $\{\Phi_\lambda\}$ if $E\epsilon\mathfrak{M}$ and $M(E)=0$ imply that $\Phi_\lambda(E)=0$ for all $\lambda\epsilon\Lambda$. (3) The family $\{\Phi_\lambda\}$ enjoys the property of equi-continuity with respect to the basis M if $\lim_{n\to\infty}M(E_n)=0$ implies $\lim_{n\to\infty}\Phi_\lambda(E_n)=0$ uniformly for all $\lambda\epsilon\Lambda$.

The following theorems are proved or stated. (1) A uniformly additive family of completely additive set functions on $\mathfrak{M}$ has a basis. (2) A family of completely additive set functions $\{\Phi_\lambda\}$, $\lambda\epsilon\Lambda$, defined on $\mathfrak{M}$ possesses a basis if and only if there exists no uncountable family $\{E_\alpha\}$, $\alpha\epsilon A$, contained in $\mathfrak{M}$, consisting of pairwise disjoint sets, on each of which some function Φ_λ is different from zero. (3) The properties of uniform additivity and equi-continuity are equivalent. (4) If $\{\Phi_\lambda\}$, $\lambda\epsilon\Lambda$, enjoys the property of uniform additivity, then the family $\{\overline{\Phi}_\lambda\}$, $\lambda\epsilon\Lambda$, also enjoys this property, where $\overline{\Phi}_\lambda(E)$ is the total variation of Φ_λ on the set E. Finally, a simple example is given which shows that the existence of a basis does not imply uniform additivity.

E. Hewitt (Chicago, Ill.).

Choquet, Gustave. Sur des ensembles cartésiens paradoxaux et la théorie de la mesure. Bull. Soc. Math. France 74, 15–25 (1946).

This paper is concerned with constructing subsets A of the Euclidean plane such that the sets $A'\cap E[(x,y),y=\mu]$ have linear measure zero and such that, for sets L in certain classes $\mathfrak{C}$, $A\cap L$ has linear measure zero provided that $L\cap E[(x,y),y=\nu]$ has linear measure zero (μ and ν are arbitrary real numbers). The author proves, under the assumption that $2^{\aleph_0}=\aleph_1$: (1) if $\mathfrak{C}$ is the class of all linearly measurable sets, then no set A as described can exist; (2) if $\mathfrak{C}$ is the class of all projective, analytic or Borel sets, then a set A as described does exist; (3) if $\mathfrak{C}$ is the class of linearly measurable sets of finite measure, then a set A as described exists. The proofs involve well-ordering and are of standard type. *E. Hewitt* (Chicago, Ill.).

Dvoretzky, A. A note on Hausdorff dimension functions. Proc. Cambridge Philos. Soc. 44, 13–16 (1948).

Given a Hausdorff measure-function $h(x)$ [Math. Ann. 79, 157–179 (1918)], it is shown that a necessary and sufficient condition that a linear set exist of finite positive h-measure is that $\liminf_{x\to 0} h(x)/x>0$. Sufficient but not necessary conditions were given by Hausdorff [loc. cit.]. The corresponding criterion in q dimensions, $\liminf_{x\to 0} h(x)/x^q>0$, is stated, the same argument applying. *H. D. Ursell.*

Hartman, S. Sur deux notions de fonctions indépendantes. Colloquium Math. 1, 19–22 (1947).

The author considers two possible definitions of pairs of independent Lebesgue measurable functions f, g defined in the unit interval. [He considers sets of more than two functions, but the result is stated for two here to simplify the notation.] The functions are independent if for sets E_1 and E_2 of some class (1) $|f^{-1}(E_1)g^{-1}(E_2)|=|f^{-1}(E_1)|\cdot|g^{-1}(E_2)|$, where $|\ |$ designates Lebesgue measure. They are independent (S) if (1) is true whenever E_1 and E_2 are intervals, independent (K) if (1) is true whenever $f^{-1}(E_1)$ and $g^{-1}(E_2)$ are Lebesgue measurable. The author shows that the two definitions are equivalent. The question proposed by Marczewski whether the definitions remain equivalent if Lebesgue measure is replaced by any measure in an abstract space remains open. *J. L. Doob* (Urbana, Ill.).

Robbins, H. E. A note on the Riemann integral. Amer. Math. Monthly 50, 617–618 (1943).

The author proves the following theorem. Let $f(x)$ be continuous on the interval $a\leqq x\leqq b$ and let C be an arbitrary but fixed constant such that $C\geqq b-a$. Choose a sequence of numbers $x_0,\cdots,x_n$ (not necessarily increasing) such that $x_0=a$, $x_n=b$, and $\sum|x_i-x_{i-1}|\leqq C$. Set $\max|x_i-x_{i-1}|=\delta_n$ and form the sum $S_n=\sum f(\bar{x}_i)(x_i-x_{i-1})$, where $\bar{x}_i$ is any point in the closed interval whose endpoints are x_i and x_{i-1}. Then $\lim S_n=\int_a^b f(x)dx$ for any sequence of sums for which $\delta_n\to 0$, the integral being the ordinary Riemann integral.

G. B. Price (Lawrence, Kan.).

Robbins, Herbert. Convergence of distributions. Ann. Math. Statistics 19, 72–76 (1948).

If $\{f_n\}$ is a sequence of probability densities, $n=0,1,2,\cdots$, then a necessary and sufficient condition that $\lim_n\int_E f_n dx=\int_E f_0 dx$ for every Borel set E is that the integrals $\int_E f_n dx$ are uniformly absolutely continuous and that $\lim_n\int_{-\infty}^{t} f_n dx=\int_{-\infty}^{t} f_0 dx$ for every real number t. Examples are given to show that various related methods of convergence for probability distributions are not equivalent.

P. R. Halmos (Princeton, N. J.).

Dienes, Paul. Sur l'intégrale de Riemann-Stieltjes. Revue Sci. 85, 259–274 (1947).

Il s'agit de l'intégrale de Stieltjes d'une fonction d'une variable réelle par rapport à une fonction d'intervalle bornée, additive, mais non nécessairement complètement additive (fonction a.b.). Toute fonction a.b. est la somme d'une fonction complètement additive (elle-même somme d'une fonction singulière A et d'une fonction continue C) et d'une fonction B dont la valeur est concentrée autour de ses points singuliers de sorte qu'au point singulier x, sa limite pour l'intervalle ouvert $(x,x+h)$ (ou $(x-h,x)$) quand $h\to 0$ n'est pas nulle. D'une fonction de point f à variation bornée, on déduit 9 fonctions a.b. différentes, selon que pour chacune des extrémités x d'un intervalle ouvert, on prend $f(x)$, $f(x^-)$

ou $f(x^+)$. Seule la fonction usuelle utilisée dans l'intégration de Stieltjes-Lebesgue est complètement additive.

Pour définir l'intégrale (de Riemann-Stieltjes) d'une fonction de point f par rapport à une fonction a.b. soit $\Gamma=A+B+C$; l'intervalle d'intégration est partagé d'une manière non stricte, certains points de division pouvant appartenir à deux intervalles ou bien à aucun. L'auteur démontre de nombreux résultats sur les limites des sommes $\underline{S}$ et $\bar{S}$ généralisant le théorème de Darboux, selon les hypothèses faites sur les partages successifs et sur A, B, C. Citons le résultat suivant: désignons par L^- et L^+ les ensembles limites de $\underline{S}$ et $\bar{S}$ quand le plus grand des intervalles tend vers zéro, les suites de divisions contenant chacun des points singuliers de Γ à partir d'un certain rang. Si on suppose $A=0$, il n'y a pas recouvrement de L^- et L^+.

L'auteur donne ensuite plusieurs définitions d'une fonction intégrable, de manière à obtenir une intégrale qui soit fonction additive, ou complètement additive, de l'intervalle d'intégration. Il indique enfin diverses conditions d'intégrabilité. De nombreux exemples sont répartis dans le mémoire.

R. de Possel (Alger).

Fréchet, M. Sur diverses définitions de l'aire. Revista Acad. Ci. Madrid **36**, 50–53 (1942).

Summary of a lecture.

Tolstov, G. P. On certain properties of partial derivatives. Doklady Akad. Nauk SSSR (N.S.) **58**, 749–751 (1947). (Russian)

This note states some properties of $F(x, y)$, $a\leqq x\leqq b$, $c\leqq y\leqq d$, which depend on the finiteness of its partial derivatives. Typical results are as follows. (1) If $F(x, y)$ possesses all possible partial derivatives of all orders up to and including the mth, then all of these partial derivatives are functions of the first Baire class. (2) Under the same hypotheses, a mixed partial derivative of order less than m is everywhere independent of the sequence in which the partial differentiations are carried out, but this need be true only almost everywhere for the mth order derivatives. It is noted that theorems of the second type generalize to n variables, while theorems of the first type do not.

M. M. Day.

Theory of Functions of Complex Variables

Ríos, Sixto. Lectures on the analytic representation of functions. Revista Acad. Ci. Madrid **38**, 287–330, 463–507 (1944); **39**, 273–319 (1945). (Spanish)

The lectures were reprinted as a book [Madrid, 1945; these Rev. **7**, 200].

Chow, Hung Ching. On the summability of a power series. Acad. Sinica Science Record **2**, 20–21 (1947).

It was proved by M. Riesz [Ark. Mat. Astr. Fys. **11**, no. 12 (1916)] that if $\alpha\geqq 0$ and $a_n=O(n^\alpha)$, then the series $\sum a_n z^n$ is bounded C_α at each point on the circle $|z|=1$ at which the analytic function $f(z)$ determined by $\sum a_n z^n$ is regular. Using this, and some facts about absolute Cesàro summability [which are used with neither statements nor references; convenient statements and references are given by L. S. Bosanquet, J. London Math. Soc. **11**, 11–15 (1936)], the author proves the following theorem. If $\alpha\geqq -1$, if $\delta>0$, and if $a_n=O(n^\alpha)$, then the series $\sum a_n z^n$ is summable $|C_{\alpha+1+\delta}|$ at each point on the circle $|z|=1$ at which the analytic function $f(z)$ defined by $\sum a_n z^n$ is regular.

There is a disturbing erratum in the proof of the lemma: if $\alpha\geqq 0$, if $\delta>0$, and if $\sum na_n$ is bounded C_α, then the series $\sum a_n$ is summable $|C_\alpha|$. The sequence $t_n^{\alpha+\delta}$ used by the author is (and should be to provide an essential part of the proof) the $C_{\alpha+\delta}$ transform of the sequence na_n; not the $C_{\alpha+\delta-1}$ transform of the series $\sum na_n$ as the author says.

R. P. Agnew (Ithaca, N. Y.).

Turán, P. On the gap-theorem of Fabry. Hungarica Acta Math. **1**, 21–29 (1947).

Let $f(z)=\sum a_n z^{\lambda_n}$, where λ_n, $n=1, 2, \cdots$, are positive integers such that $n/\lambda_n\to 0$ as $n\to\infty$. Then the Fabry theorem asserts that $f(z)$ has the circumference of its circle of convergence as a natural boundary. The author proves this by use of the inequality

$$\max_{0\leqq x\leqq 2\pi}\left|\sum_{r=1}^{n} a_r e^{i\lambda_r x}\right|\leqq(48\pi/\delta)^n \max_{a\leqq x\leqq a+\delta}\left|\sum_{r=1}^{n} a_r e^{i\lambda_r x}\right|.$$

N. Levinson (Cambridge, Mass.).

Cowling, V. F. Some results for Dirichlet series. Duke Math. J. **14**, 907–911 (1947).

Let $h>0$, and $0<\beta\leqq\pi$. Let $a(w)$ be regular in the region $|\arg(w-h)|\leqq\beta$, and there satisfy the condition $|a(h+re^{i\psi})|\leqq r^k\exp(-Lr\sin\psi)$ for some k and some L, $0<L<2\pi$, and all large r. Then $f(z)=\sum_1^\infty a(n)n^{-z}$ is entire, if the series converges for some z.

R. C. Buck.

Eweida, M. T. Order of magnitude of the zeros of polynomials in basic series. Duke Math. J. **14**, 865–875 (1947).

Let $\{p_n(z)\}$ be a simple basic set of polynomials [J. M. Whittaker, Interpolatory Function Theory, Cambridge University Press, 1935], of form $p_0(z)=1$; $p_n(z)=\prod_{r=1}^{n}(z-a_{n,r})$, $n>0$. Extending and improving on results of M. Nassif [Proc. Math. Phys. Soc. Egypt **2**, 1–6 (1944); these Rev. **7**, 425], it is shown that: (I) if $|a_{n,r}|\leqq k/n^\alpha$, with $0\leqq\alpha<1$, then the basic series determined by $\{p_n(z)\}$ represents all integral functions of increase less than order $1/(1-\alpha)$, type $(1-\alpha)\{k^{-1}\log 2\}^{1/(1-\alpha)}$ in every bounded region; (II) if $|a_{n,r}|\leqq k/n$, then the basic series is effective in $|z|\leqq R$ for all $R\geqq k/\log 2$ (i.e., if $f(z)$ is regular in $|z|\leqq R$ for such an R, the basic series converges uniformly to $f(z)$ in $|z|\leqq R$). Some examples are given by way of illustration.

I. M. Sheffer (State College, Pa.).

Bose, S. K. On the derivatives of integral functions. J. Indian Math. Soc. (N.S.) **10**, 77–80 (1946).

La fonction $f(z)$ étant entière, $M(r)$ le maximum de son module pour $|z|=r$, $M^p(r)$ le maximum du module de la dérivée d'ordre p, l'auteur compare les fonctions $M^p(r)$. Il montre notamment que, p étant donné arbitraire, on a $M(r)>M^1(r)>\cdots>M^p(r)$ pour $r>r_0(f)$ pourvu que $f(z)$ soit d'ordre inférieur supérieur à 1, tandis que $M(r)<M^1(r)<\cdots<M^p(r)$ pour $r>r_0(f)$ si l'ordre ρ est inférieur à $\frac{1}{2}$. La démonstration de la première proposition repose [p. 78] sur une égalité asymptotique qui n'est vraie qu'à l'extérieur de certains intervalles; il faudrait la compléter. Dans la seconde proposition, l'emploi de l'inégalité $M^1(r)<r^{\rho-1+\epsilon}M(r)$ [voir Valiron, Enseignment Math. **28**, 55–59 (1929)] montre que la limite $\frac{1}{2}$ peut être remplacée par 1.

G. Valiron (Paris).

Chuang, Chi-Tai. Étude sur les familles normales et les familles quasi-normales de fonctions méromorphes. Rend. Circ. Mat. Palermo 62, 1–80 (1939).

The author studies generalizations of a problem proposed by Montel concerning conditions of normality of a family of analytic functions $F(z)$, such that in the domain D of definition $F(z)$ does not assume the value 0, and for a fixed positive integer n the derivative $F^{(n)}(z)$ does not assume the value 1. If we write $F(z)=e^{f(z)}$, the equation $F^{(n)}(z)=1$ takes the form (1) $e^f[f'^n+P(f',f'',\cdots,f^{(n)})]=1$, where P is a polynomial of degree $n-1$. In place of (1) the author considers an equation of the form

$$\Pi(f,f',\cdots,f^{(k)})+P(f,f',\cdots,f^{(p)})=1, \tag{2}$$

where Π is a monomial and P a polynomial, or of the form

$$\sum_{j=1}^{n} a_j(z)f^{(n-j)}(z)=1, \tag{3}$$

where the $a_j(z)$ are given analytic functions with $a_0(z)\neq 0$ in D; he investigates families of functions $f(z)$ which are analytic, nonvanishing and such that (2) or (3) has no root in D. Results analogous to known theorems of Valiron, Miranda, Bloch and others are obtained. In a second part the author studies families of meromorphic functions, obtaining criteria for quasi-normal families and generalizing theorems of Ahlfors. *E. F. Beckenbach.*

Lokki, Olli. Über analytische Funktionen, deren Dirichlet-integral endlich ist und die in gegebenen Punkten vorgeschriebene Werte annehmen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 39, 57 pp. (1947).

The author considers the class E of functions

$$w(z)=\sum_{m=0}^{\infty} a_m z^m$$

for which

$$I=\pi^{-1}\int_0^1\int_0^{2\pi}|w'(z)|^2 r\,dr\,d\phi=\sum_{m=1}^{\infty} m|a_m|^2<\infty.$$

Denoting by E_n the subclass of E such that $w(z_\nu)=w_\nu$, $\nu=1,\cdots,n$, he obtains first the unique extremal function $w_n(z)$ of E_n, which gives to I its minimum I_n, as previously found by Takenaka [Tôhoku Math. J. 27, 21–40 (1926)]. He also shows that the values w which a function of E takes at a point z_0 different from the z_ν, when I takes a given value $I_0>I_n$, fill a circle $|w-w_0|^2\leqq k(I_0-I_n)$, where $k>0$ depends only on the z_ν and on z_0 and $w_0=w_n(z_0)$. The subclass E_∞ of all the $w(z)$ for which $w(z_\nu)=w_\nu$, $1\leqq\nu<\infty$, is shown to be nonempty if and only if $I_n\to I_\infty<\infty$ as $n\to\infty$. If further $\sum(1-|z_\nu|)$ converges the results obtained for E_n extend to E_∞, but if $\sum(1-|z_\nu|)$ diverges, E_∞ contains at most one member. *W. K. Hayman* (Exeter).

Kufarev, P. P. On a method of numerical determination of the parameters in the Schwarz-Christoffel integral. Doklady Akad. Nauk SSSR (N.S.) 57, 535–537 (1947). (Russian)

The integral referred to in the title is that concerned with the conformal representation of the interior of a circle on a simple connected domain bounded by a polygon. The method is developed in the present paper for the special case in which the polygonal domain consists of the whole plane cut by a broken line having a finite number of sides one of which extends to infinity. Denoting the vertices of this polygon in order by $\infty, A_0, A_1, A_2, \cdots$, the whole z-plane cut by the polygon $\infty, A_0, \cdots, A_k$ can be represented on $|w|<1$ by a function $z=f_k(w)$ expressible as a Riemann-Christoffel integral. Since $f_0(w)$ is an elementary function, the problem will be solved if $f_{k+1}(w)$ can be derived from $f_k(w)$. Let A be an arbitrary point on the segment A_kA_{k+1}; then the plane cut by $\infty, A_0, A_1, A_2, \cdots, A_k, A$ will be represented on $|w|<1$ by a function of the form

$$z=f(w,t)=e^{-t}\int_0^w \frac{1-w/\mu(t)}{(1-w/a(t))^3}\prod_{p=0}^{k}\left(\frac{1-w/a_p(t)}{1-w/b_p(t)}\right)^{x_p}dw,$$

where $\mu(t)$ and $a(t)$ are the points on $|w|=1$ which correspond to $z=A$ and $z=\infty$, and $a_p(t)$, $b_p(t)$ are the two points on $|w|=1$ which correspond to the vertex A_p. We shall have $f_w(0,t)=e^{-t}$, so that, as A varies from A_k to A_{k+1}, t will increase from say t_k to t_{k+1} and $f(w,t)$ will change continuously from $f_k(w)$ to $f_{k+1}(w)$. Then $f(w,t)$ will satisfy the equation of Löwner [Math. Ann. 89, 103–121 (1923)]:

$$\frac{\partial f}{\partial t}+w\frac{\mu(t)+w}{\mu(t)-w}\frac{\partial f}{\partial w}=0.$$

Setting

$$\mu(t)=e^{i\lambda(t)},\quad a(t)=e^{i\varphi(t)},\quad a_p(t)=e^{i\alpha_p t},\quad b_p(t)=e^{i\beta_p(t)},$$

there results the system of equations

$$d\alpha_p/dt=\cot\tfrac{1}{2}(\alpha_p-\lambda),\quad d\beta_p/dt=\cot\tfrac{1}{2}(\beta_p-\lambda),$$
$$d\varphi/dt=\cot\tfrac{1}{2}(\varphi-\lambda),$$
$$\frac{d\lambda}{dt}=3\frac{d\varphi}{dt}+\sum_0^k x_p\left(\frac{d\beta_p}{dt}-\frac{d\alpha_p}{dt}\right).$$

This system possesses a unique solution such that for $t=t_k$ the parameters $\mu(t)$, $a(t)$, $\alpha_p(t)$, $\beta_p(t)$ take the values (assumed known) corresponding to the polygon with $A=A_k$. There only remains the determination of the value of t_{k+1}. An equation for this is obtained by considering that if s represents the length A_kA_{k+1} then $\int_{t_k}^{t_{k+1}}ds$ is equal to the length A_kA_{k+1}, where

$$\frac{ds}{dt}=\tfrac{1}{2}e^{-t}\left|\sin^{-3}(\lambda-\varphi)\prod_0^k\left(\frac{\sin(\lambda-\alpha_p)}{\sin(\lambda-\beta_p)}\right)^{x_p}\right|.$$

A. J. Macintyre (Aberdeen).

Wright, E. M. Iteration of the exponential functions. Quart. J. Math., Oxford Ser. 18, 228–235 (1947).

The iteration theory of an analytic function $f(x)$ is closely related to Schroeder's functional equation

$$\psi\{f(x)\}=\alpha\psi(x). \tag{1}$$

Let $x=\zeta$ satisfy the equation $f(x)=x$, so that ζ is a double point of the iteration; let $f'(\zeta)=a$. Koenigs showed [Ann. Sci. École Norm. Sup. (3) 1, Suppl., 3–41 (1884); 2, 385–404 (1885)] that, if $0<|a|<1$ (case of an attractive double point), then for $\alpha=a$ equation (1) has the solution $\psi(x)=y(x)=\lim_{n\to\infty}a^{-n}\{f_n(x)-\zeta\}$ (f_n the nth iterate of $f(x)$), with $y(x)$ regular in some neighborhood of $x=\zeta$; and $y(\zeta)=0$, $y'(\zeta)=1$. If $|a|>1$ (case of a repulsive double point), the above results can be applied to the inverse function of $f(x)$. In either case (attractive or repulsive double point), $y(x)$ has an inverse function $x=X(y)$ that is regular in some neighborhood of $y=0$.

The present article deals principally with the case $f(x)=ce^x$, $c>0$. There are infinitely many double points. Writing $X(y)=\sum_0^\infty a_m y^m$ ($a_0=\zeta$, $a_1=1$), the following is shown. (I) If ζ is a repulsive double point, then $X(y)$ is an entire function, and the coefficients $\{a_m\}$ satisfy the relations

$ma_m(1-\zeta^{1-m})=\sum_{p=1}^{m-1}pa_pa_{m-p}\zeta^{-p}$ $(m\geqq 2)$; $|a_m|<K^m|\zeta|^{-mj(m)}$, where K is independent of m and $j(m)$ is defined by $j(t)=0$ $(1\leqq t<e)$; $j(t)=j(\log t)+1$ $(t\geqq e)$. (II) If $0<c<e^{-1}$ there is precisely one attractive double point ξ, and ξ is real.If $X(y)$ is associated with this attractive point ξ, then results analogous to those of (I) above are obtained.

I. M. Sheffer (State College, Pa.).

Teixidor, J. On the representation of a complex S_2 by means of a real S_4. Revista Mat. Hisp.-Amer. (4) 7, 173–177 (1947). (Spanish)

Eine elementare Beschreibung der Korrespondenz unendlichferner Punkte bei der Abbildung des Raumes zweier komplexer Veränderlichen auf einen reellen vierdimensionalen Raum.

P. Thullen (Bogotá).

Theory of Series

Herreilers, H. An elementary derivation of the sum of a hyperharmonic series with even exponent. Nieuw Tijdschr. Wiskunde 35, 179–188 (1947). (Dutch)

Araujo, Roberto. Limit of a sum of infinitely many variable terms. Revista Acad. Ci. Zaragoza (2) 2, 54–57 (1947). (Spanish)

Dvoretzky, Aryeh. On monotone series. Amer. J. Math. 70, 167–173 (1948).

Let c_n be the general term of a monotone convergent series and d_n of a monotone divergent series. R. W. Hamming [Amer. J. Math. 68, 133–136 (1946); these Rev. 7, 292] has recently made a study of the possible occurrence of the inequality $c_n\geqq d_n$. The present paper consists of a study, with certain generalizations. of Hamming's results.

T. Fort (Athens, Ga.).

Szász, Otto. Quasi-monotone series. Amer. J. Math. 70, 203–206 (1948).

In this paper the familiar requirement, $a_n\geqq a_{n+1}>0$, is replaced by $0<a_{n+1}\leqq a_n(1+\alpha/n)$. The author generalizes the "Cauchy condensation test" for convergence to apply to series of this type. He also generalizes the "Cauchy integral test" where the assumptions are $0<a(x+y)\leqq(1+\alpha/x)a(x)$ for $x>0$ and $0<y<1$. It is then proved that if $a(x)$ is Riemann integrable, $h\sum_{\nu=1}^{\infty}a(\nu h)\rightarrow\int_0^{\infty}a(x)dx$.

T. Fort.

Henstock, R. The efficiency of matrices for Taylor series. J. London Math. Soc. 22, 104–107 (1947).

Let $g_m(z)=\sum_{n=0}^{\infty}a_{mn}s_n(z)$ be a sequence-to-sequence transformation by which a series $\sum u_n(z)$ with partial sums $s_n(z)$ is summable (or uniformly summable) over a set E to $f(z)$ if $g_m(z)\rightarrow f(z)$ (or $g_m(z)\rightarrow f(z)$ uniformly) over E. Let the matrix a_{mn} be such that, whenever $\sum c_nz^n$ has a positive radius of convergence and E is a bounded closed set in the Mittag-Leffler star of the function $f(z)$ generated by $\sum a_nz^n$, the series is uniformly summable over E to $f(z)$. Such matrices have been characterized by Borel [Leçons sur les Séries Divergentes, Paris, 1901, pp. 164 ff.] and Okada [Math. Z. 23, 62–71 (1925)]. The author shows how to modify the matrix a_{mn} to obtain a new matrix c_{mn} which also has the same effectiveness inside Mittag-Leffler stars, but which nevertheless violates the norm condition of the Silverman-Toeplitz theorem and hence is not regular. The paper treats "partial star domains" which are Mittag-Leffler stars or special subsets of them.

R. P. Agnew.

Rajagopal, C. T. On Riesz summability and summability by Dirichlet's series. Amer. J. Math. 69, 851–852 (1947).

The author gives a criterion for the summability (R, λ, k), $k\geqq 0$, of a Dirichlet series $\sum a_ne^{-\lambda_n s}$, the statement of which he attributes to the reviewer. This replaces corollary 2 of his paper under the same title [Amer. J. Math. 69, 371–378 (1947); these Rev. 9, 26], in which he assumed that the summability (R, λ, k) of $\sum a_n$ implies the convergence of

$$\int_0^{\infty}\Big(\sum_{\lambda_n\leqq t}a_n\Big)e^{-\epsilon t}dt$$

for every $\epsilon>0$, whereas this is not so when $k>1$, e.g., if $a_n=(-1)^nn^2$, $\lambda_n=\log n$, $k=2$.

L. S. Bosanquet (London).

Lorentz, G. G. Über Limitierungsverfahren, die von einem Stieltjes-Integral abhängen. Acta Math. 79, 255–272 (1947).

A function $a(x, t)$, defined for $x, t\geqq 0$, is said to determine a transformation

$$\sigma(x)=\int_0^{\infty}a(x, t)ds(t)=\lim_{c\to\infty}\int_0^c a(x, t)ds(t) \tag{1}$$

by which $s(t)$ is assigned the generalized limit $\sigma=L_A(s)$ if $\sigma(x)$ exists for each $x\geqq 0$ and $\sigma(x)\rightarrow\sigma$ as $x\rightarrow\infty$. The last integral in (1) is a Riemann-Stieltjes integral. The transformation (1) is conservative (konvergenztreu) over a class E of functions $s(t)$ if $\sigma=L_A(s)$ exists whenever $s(t)$ belongs to E and $\lim_{t\to\infty}s(t)$ exists, and is regular (permanent) over E if $L_A(s)=\lim_{t\to\infty}s(t)$ whenever $s(t)$ belongs to E and $\lim_{t\to\infty}s(t)$ exists. Three conditions, of which one is the existence of $a^*(t)=\lim_{x\to\infty}a(x, t)$, $t\geqq 0$, characterize the transformations (1) conservative over the class E_v of functions $s(t)$ having bounded variation over each finite interval $0\leqq t\leqq c$. For regularity over the subclass E_v', containing those functions $s(t)$ in E_v for which $s(0)=0$, the same three conditions, with $a^*(t)\equiv 1$, are necessary and sufficient. With some restrictions on the functions involved, the "dual" transformation

$$\tau(x)=-\int_0^{\infty}s(t)da(x, t)$$

is studied and compared with (1). There are applications and examples involving inclusion relations among transformations of the type

$$\sigma(x)=\int_0^x p(t)s(t)dt\Big/\int_0^x p(t)dt,$$

and among matrix transformations of series and sequences.

R. P. Agnew (Ithaca, N. Y.).

Ogieveckiĭ, I. I. An extension of the theorem of Frobenius to double power series. Doklady Akad. Nauk SSSR (N.S.) 58, 1897–1900 (1947). (Russian)

A double series $\sum a_{mn}$ with partial sums s_{mn} is summable to L by the Cesàro method $C_{1,1}$ if $\sigma_{mn}\rightarrow L$ as $mn\rightarrow\infty$, and by the Euler-Abel power series method if $f(x, y)\rightarrow L$ as $x, y\rightarrow 1^-$, where

$$\sigma_{mn}=(m+1)^{-1}(n+1)^{-1}\sum_{j,k=0}^{m,n}s_{jk};\quad f(x, y)=\sum_{j,k=0}^{\infty}s_{jk}x^jy^k.$$

It is well known that some convergent sequences having unbounded partial sums fail to be summable $C_{1,1}$; and some summable $C_{1,1}$ fail to be summable by the power series

method. It is shown that if $\sum u_{mn}$ is summable $C_{1,1}$ to L and if the partial sums are such that $\limsup_{m\to\infty} |s_{mn}|/m < \infty$, $\limsup_{n\to\infty} |s_{mn}|/n < \infty$ for each n and m, respectively, then, for each $\lambda > 1$, $f(x, y) \to L$ as $x, y \to 1$ over $0 < x < 1$, $0 < y < 1$ subject to the restriction $\lambda^{-1} \leqq (1-x)/(1-y) \leqq \lambda$. For a more extensive treatment of this subject, see Vignaux [An. Soc. Ci. Argentina 126, 321–344, 401–428 (1938); 127, 161–185 (1939)] and references given there. See also Vignaux [Boll. Un. Mat. Ital. (1) 17, 209–214 (1938)]. The reviewer should have called attention to these references in his review [these Rev. 9, 87] of a related paper of Chelidze [C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 691–694 (1946)] on power-series summability of convergent double series.

R. P. Agnew (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Loo, Ching-Tsün. Note on the strong summability of Fourier series. Duke Math. J. 14, 913–919 (1947).

Suppose $f(x)$ is of period 2π and $f(x) \varepsilon L^p(-\pi, \pi)$, $p > 1$. If $S_n = S_n(x)$ denotes the nth partial sum of the Fourier series of f, and $\varphi(u) = \frac{1}{2}\{f(x+u) + f(x-u) - 2c\}$, then the condition $\int_0^t |\varphi(u)|^p du = o(t)$ as $t \to +0$ implies $\sum_{m=0}^{n} |S_{m(k)} - c|^2 = o(n)$ as $n \to \infty$, where $m(k) = m^k$, for every integer $k \geqq 2$. This is an improvement on a previous result of the author [Trans. Amer. Math. Soc. 56, 519–527 (1944); these Rev. 6, 126].

K. Chandrasekharan (Princeton, N. J.).

Cheng, Min-Teh. Summability of Hardy's associate series of a Fourier series. Acad. Sinica Science Record 2, 39–44 (1947).

J. M. Hyslop and H. C. Chow (independently for $0 < \alpha \leqq \frac{1}{2}$) and K.-K. Chen (for $\frac{1}{2} < \alpha < 1$) proved that, if $f(x)$ belongs to Lip α $(0 < \alpha < 1)$ in $(0, \pi)$, then its Fourier series is summable $|C, \beta|$ for $\beta > \frac{1}{2} - \alpha$ [Proc. London Math. Soc. (2) 43, 475–483, 484–489 (1937); Amer. J. Math. 66, 299–312 (1944); these Rev. 5, 262]. Here the author proves further (in the case $0 < \alpha \leqq \frac{1}{2}$) that $\sum (a_n \cos nx + b_n \sin nx)/(\log n)^{1+\varepsilon}$ is summable $|C, \frac{1}{2} - \alpha|$, and that ε cannot be replaced by zero.

L. S. Bosanquet (London).

Misra, M. L. The summability (A) of the conjugate series of a Fourier series. Duke Math. J. 14, 855–863 (1947).

Let $f(x)$ be Lebesgue integrable and of period 2π,

$$\psi(t) = \psi_0(t) = f(x+t) - f(x-t),$$

and

$$\psi_n(t) = t^{-1} \int_0^t \psi_{n-1}(t) dt.$$

At any point x at which $\psi_{n+1}(t) = o(1)$ as $t \to 0$, where n in a nonnegative integer, a necessary and sufficient condition for the Abel summability of the conjugate Fourier series of $f(x)$ is that $\pi^{-1} \int_0^\infty \psi(t) t^{-1} dt$ exist (C, n). The value of the latter integral will in turn be the Abel sum of the conjugate series.

P. Civin (Eugene, Ore.).

Chandrasekharan, K., and Minakshisundaram, S. Some results on double Fourier series. Duke Math. J. 14, 731–753 (1947).

The spherical partial sum of order δ for a periodic function $f(x, y)$ of class L at a given point (x^0, y^0) is

$$(*) \quad S_R^\delta(x^0, y^0) = \sum_{n^2+m^2 \leqq R^2} (1 - (n^2+m^2)/R^2)^\delta c_{mn} e^{i(mx^0+ny^0)}.$$

For $\delta > \frac{1}{2}$ it is known to converge to $f(x^0, y^0)$, as $R \to \infty$, for very general continuity behavior of $f(x, y)$ at (x^0, y^0). Now the authors show first of all that if $f(x, y)$ is literally continuous at the point, then $S_R^\delta(x)$ is (continuously) convergent for the double limit $\{R \to \infty; (x, y) \to (x^0, y^0)\}$. Furthermore, they give a (Tauberian-like) convergence test for $\delta = 0$, of the Hardy-Littlewood kind, which explicitly restricts the magnitude of $|c_{m,n}|$ as $m^2 + n^2 \to \infty$; and on the other hand they give for any $\delta \geqq 0$ statements on the degree of convergence of (*) towards $f(x^0, y^0)$. Finally, they simplify the reasoning underlying a recent paper on absolute convergence by Minakshisundaram and Szász [Trans. Amer. Math. Soc. 61, 36–53 (1947); these Rev. 8, 376].

S. Bochner (Princeton, N. J.).

Wintner, Aurel. On Töpler's wave analysis. Amer. J. Math. 69, 758–768 (1947).

The author studies the completeness of sets of functions $\{\phi(nt)\}$ by the methods of Toeplitz matrices and Dirichlet series. Similar results have already been obtained by similar methods by Bourgin [Trans. Amer. Math. Soc. 60, 478–518 (1946); these Rev. 8, 512].

H. Pollard (Ithaca, N. Y.).

Wintner, Aurel. The sum formula of Euler-Maclaurin and the inversions of Fourier and Möbius. Amer. J. Math. 69, 685–708 (1947).

The author investigates the approximation of improper integrals by equidistant Riemann sums. The most striking of the results is the following, which is deeper than the prime number theorem, and in fact contains the theorem that Lambert summability implies Abel summability. If $f(x)$ is continuous on $0 < x \leqq 1$ and the limit

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{n\varepsilon \leqq 1} f(n\varepsilon)$$

exists, then $\int_{0+}^1 f(x) dx$ exists and has the same value. This is closely related to the inversions mentioned in the title, and to Poisson's summation formula in the theory of Fourier transforms. The exact connections are carefully examined.

As a consequence of his results the author obtains a formula for the stable distribution corresponding to the function $e^{-|x|^\lambda}$, $0 < \lambda < 1$.

H. Pollard (Ithaca, N. Y.).

Guinand, A. P. Some formulae for the Riemann zeta-function. J. London Math. Soc. 22, 14–18 (1947).

The author shows that if $\phi(x)$ is absolutely continuous on every finite interval on $x > 0$, tends to 0 at ∞, and $x\phi'(x)$ is of class $L^2(0, \infty)$, then for $x > 0$ the limits

$$f(x) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \phi(n/x) - \int_0^N \phi(t/x) dt \right\} / x,$$

$$g(x) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \phi(x/n)/n - \int^N \phi(x/t) dt \right\}$$

exist and are a pair of Fourier transforms of class $L^2(0, \infty)$ with respect to the kernel $2 \cos 2\pi x$. It is of interest to compare this result with those of Duffin [Bull. Amer. Math. Soc. 51, 447–455 (1945); these Rev. 6, 266] and of Wintner [see the paper reviewed above, pp. 695 ff.]. If $\phi(x)$ has the additional property that f, g are so smooth (locally) as to allow the application of the Fourier integral formula, there result formulae involving the Riemann zeta function.

P. Hartman (Baltimore, Md.).

Tagamlitzki, Yaroslav. Sur la majoration de certaines transformées intégrales. C. R. Acad. Sci. Paris **225**, 1053–1055 (1947).

Let $f(x)$ be defined by the Laplace integral $\int_0^\infty e^{-xt}d\alpha(t)$ and define the divided differences

$$N[\alpha(t), t_1, \cdots, t_m] = \sum_{i=1}^{m} \alpha(t_i)/F'(t_i), \quad F(t) = (t-t_1) \cdots (t-t_m).$$

The following theorem is stated, with an indication of the proof. In order that the differences $N[\alpha(t), t_1, \cdots, t_{r+1}]$, $r=1, 2, \cdots, n-1$, be of fixed sign for fixed r, it is necessary and sufficient that $(-1)^{k+r-1}d^{k+r-1}[x^{r-1}f(x)]/dx^{k+r-1}$, $k=0, 1, 2, \cdots$, have the same sign as N, when $k, t_1, \cdots, t_{r+1}$ vary. For $r=1$ this is the familiar Hausdorff-Bernstein-Widder theorem on completely monotonic functions. The technique is that used by Widder [Trans. Amer. Math. Soc. **36**, 107–200 (1934)]. *H. Pollard* (Ithaca, N. Y.).

Erdélyi, A. Asymptotic representation of Laplace transforms with an application to inverse factorial series. Proc. Edinburgh Math. Soc. (2) **8**, 20–24 (1947).

The author proves two theorems involving the Laplace transform $\phi(z)$ of a function $f(t)$. In theorem 2, from $f(t) \sim \sum c_n g_n(t)$, as $t \to +0$, he deduces under certain general conditions $\phi(z) \sim \sum c_n \psi_n(z)$, as $|z| \to +\infty$, valid in the sector $|\arg z| \leqq \frac{1}{2}\pi - \Delta$, where $\Delta > 0$; $\psi_n(z)$ is the Laplace transform of $g_n(t)$. Special cases:

$$\text{(I)} \qquad g_n(t) = (2 \sinh \tfrac{1}{2}t)^{2n}, \quad \psi_n(z) = \frac{(2n)!\Gamma(z-n)}{\Gamma(z+n+1)};$$

$$\text{(II)} \qquad g_n(t) = (1-e^{-t})^n, \quad \psi_n(z) = \frac{n!\Gamma(z)}{\Gamma(z+n+1)};$$

$$\text{(III)} \qquad g_n(t) = (e^t-1)^n, \quad \psi_n(z) = \frac{n!\Gamma(z-n)}{\Gamma(z+1)}.$$

J. G. van der Corput (Amsterdam).

Rios, Sixto. On the singularities of the Laplace integral. Revista Acad. Ci. Madrid **36**, 119–125 (1942). (Spanish)

Cf. Portugaliae Math. **3**, 110–114 (1942); these Rev. **4**, 40.

Pollard, Harry. The mean convergence of orthogonal series. I. Trans. Amer. Math. Soc. **62**, 387–403 (1947).

Let $f(x) \varepsilon L^p(-1, 1)$, $p > 1$, and let $s_n(x)$ be the nth partial sum of the Fourier-Legendre development $a_0P_0 + a_1P_1 + \cdots$ of f. Then

$$(*) \qquad \lim_{n\to\infty} \int_{-1}^{1} |f - s_n|^p dx = 0$$

if $4/3 < p < 4$. For the values of p outside the interval $(4/3, 4)$, (*) need not hold. The cases $p = 4/3$ and $p = 4$ are still open. This is the first nontrivial extension to classical polynomials of M. Riesz's well-known results concerning mean convergence of trigonometric Fourier series [Math. Z. **27**, 218–244 (1927)]. The corresponding extensions (also considered by the author) to Sturm-Liouville developments are consequences of the equiconvergence theorems.

A. Zygmund (Chicago, Ill.).

Nikol'skiĭ, S. Best approximation in the mean of a class of functions by arbitrary polynomials. Doklady Akad. Nauk SSSR (N.S.) **58**, 25–28 (1947). (Russian)

Let $\psi_0(x), \psi_1(x), \cdots, \psi_n(x)$ be fixed functions from $L(a, b)$, and for any $f \varepsilon L(a, b)$ let

$$E_n(f)_L = \min_{\lambda_k} \int_a^b |f(x) - \sum_0^n \lambda_k \psi_k(x)| dx.$$

Let $K(t, x)$ be defined for $a \leqq t \leqq b$, $a \leqq x \leqq b$, and let it satisfy the conditions $|K(t, x)| \leqq M(x) \varepsilon L(a, b)$,

$$\lim_{t\to t} \int_a^b |K(t, x) - K(t_0, x)| dx = 0$$

for every $t_0 \varepsilon (a, b)$. Let HV denote the class of functions $f(x) = \int_a^b K(t, x) d\varphi(t)$, where φ is of bounded variation in (a, b), its total variation there being at most 1. Let $\mathcal{E}(HV)_L = \sup E_n(f)_L$ for all $f \varepsilon HV$. It is shown that (1) $\mathcal{E}(HV)_L = \max_{a \leqq t \leqq b} E_n(K(t, x))_L$, where $K_n(t, x)$ under the sign of E_n is treated as a function of x. Consider the sums $P_n(t, x) = \sum_0^n \alpha_k(t)\psi_k(x)$, where $\alpha_k(t)$ are certain functions continuous in (a, b). Then

$$U_n(f, x) = \int_a^b P_n(t, x) d\varphi(t) = \sum_0^n a_k \psi_k(x), \qquad a_k = \int_a^b \alpha_k d\varphi(t),$$

is a linear combination of the ψ's and may be treated as an approximation to f. The expression $U_n(f, x)$ will be the best method of approximating f, if

$$(2) \qquad \sup_{f \varepsilon HV} \int_a^b |f(x) - U_n(f, x)| dx = \mathcal{E}(HV)_L.$$

Let t^* be the value of t for which the maximum in (1) is reached. The author shows that a necessary and sufficient condition for (2) is

$$\int_a^b |K(t, x) - P_n(t, x)| dx \leqq E_n(K(t^*, x))_L, \qquad a \leqq t \leqq b.$$

Of interest is the special case when $(a, b) = (0, 2\pi)$, $K(t, x) = K(t-x)$ and $E_n(f)$ is the approximation in L of f by trigonometric polynomials of order $n-1$.

A. Zygmund (Chicago, Ill.).

Wilkins, J. Ernest, Jr. A note on the general summability of functions. Ann. of Math. (2) **49**, 189–199 (1948).

For a broad class of kernels, summability relations of the type

$$(*) \quad \int_a^b K(x, y, \lambda) f(y) dy = \tfrac{1}{2} \sum_{i=0}^{p} \lambda^{-i} \sum_{j=0}^{iq} a_{ij} \{ f_j^+(x) + f_j^-(x) \} = o(\lambda^{-p})$$

as $\lambda \to \infty$ are discussed. If a nonnegative function $\Phi(y)$ has a generalized derivative $\Phi_{pq}(y)$ of order pq and suitable behavior at the end points a and b and if (*) is satisfied by $f(y) = y^k \Phi(y)$ for $k = 0, 1, \cdots, pq$, then (*) will be satisfied by any function possessing one-sided generalized derivatives $f_{pq}^+(x)$ and $f_{pq}^-(x)$, and suitable behavior at the end points. The theorem includes as a special case a theorem of Titchmarsh [Introduction to the Theory of Fourier Integrals, Oxford, 1937, p. 32].

Examples are given of the application to various kernels. In particular, when the left member of (*) becomes the singular integral of de la Vallée Poussin, the complete asymptotic expansion is given for functions $f(y) \varepsilon C^\infty$. Natanson gave the first two terms of the expansion [C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 274–277 (1944); these Rev. **6**, 267] for functions $f(y) \varepsilon C^2$. *P. Civin.*

Brāzma, N. Sur les fonctions presque-périodiques de plusieurs variables complexes. Acta Univ. Latviensis [Latvijas Univ. Raksti] Ser. III. **20**, 431–455 (1941). (French. Latvian summary)

A function $f(s) = f(s_1, \cdots, s_n)$, where $s_\nu = \sigma_\nu + it_\nu$, analytic in the domain $a_\nu < \sigma_\nu < b_\nu$; $-\infty < t_\nu < \infty$; $\nu = 1, \cdots, n$, is called almost periodic in this domain if there corresponds

to any $\epsilon>0$ a real number l such that any parallelepiped $c_\nu<t_\nu<c_\nu+l$; $\nu=1, \cdots, n$ contains a point $\tau=(\tau_1, \cdots, \tau_n)$ satisfying $|f(s+\tau)-f(s)|\leqq\epsilon$ for all sets of values $s=(s_1, \cdots, s_n)$ in the domain. The author develops the theory of these almost periodic functions in close analogy with the theory developed by H. Bohr [Acta Math. **47**, 237–281 (1926)]. The Fourier series (Dirichlet series) is introduced and its summability is proved. It is proved that a function is almost periodic if it is bounded and has almost periodic partial derivatives. A theory of functions in more general domains is not attempted. *H. Tornehave.*

Polynomials, Polynomial Approximations

Marden, Morris. The number of zeros of a polynomial in a circle. Proc. Nat. Acad. Sci. U. S. A. **34**, 15–17 (1948).

The following theorem is stated, with an indication of a proof. Let $f(z)=\sum_{k=0}^n a_k z^k$, $f^*(z)=\sum_{k=0}^n \bar{a}_k z^{n-k}$, where $\bar{a}_k$ denotes the complex conjugate of a_k. Let the sequence of polynomials $f_j(z)=\sum_{k=0}^{n-j} a_k^{(n-j)} z^k$ be defined by the recurrence formula

$$f_0(z)=f(z), \quad f_{j+1}(z)=\bar{a}_0^{(j)} f_j(z)-a_{n-j}^j f_j^*(z),$$
$$j=0, 1, \cdots, n-1.$$

If $\delta_j=f_j(0)\neq 0$ for $j=1, \cdots, n$, then $f(z)$ has no roots upon the unit circle $C: |z|=1$ and the number of roots within C is equal to the number of negative products $\delta_1\delta_2 \cdots \delta_j$, $j=1, \cdots, n$. In case $\delta_j\neq 0$, $j=1, \cdots, k$; $f_{k+1}(z)\equiv 0$, $k<n$, then $f(z)$ has $n-k$ roots on C and the number of roots within C is equal to the number of negative products $\delta_1\delta_2 \cdots \delta_j$, $j=1, \cdots, k$. The theorem is applied to obtain a criterion, involving determinants, of Schur and Cohn. *H. S. Wall* (Austin, Tex.).

v. Szökefalvi Nagy, Gyula (Julius). Sätze über die Lage von Nullstellen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 1–13 (1942). (Hungarian. German summary)

The main theorem is the following. Let a_j denote a point on the real axis and C_j and K_j the circles of the respective radii R and nR drawn in the upper half-plane tangent to the real axis at a_j. Let $G(z)=f'(z)+(A+R^{-1}i)f(z)$, where A is an arbitrary real number. If $f(z)$ is a real entire function with the a_j ($j=1, 2, \cdots$) as its only zeros and with genus 0 or 1, then $G(z)$ has no imaginary zeros inside any circle C_j. If in addition $f(z)$ is a polynomial of degree n, $G(z)$ has no imaginary zeros outside of all the K_j. [Reviewer's note: the theorem may be proved by comparing the terms in $\Im(f'/f)$ with $1/R$.] *M. Marden* (Milwaukee, Wis.).

v. Szökefalvi Nagy, Gyula. Ein elementargeometrischer Satz und seine Anwendung in der Geometrie der Polynome. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 776–785 (1942). (Hungarian. German summary)

M. Biernacki proved [Bull. Int. Acad. Polonaise Sci. Lett. Cl. Sci. Math. Nat. Sér. A. **1927**, 541–685 (1928), pp. 654–660] that, if $m_1, \cdots, m_n$ are positive integers and if $z_1, \cdots, z_n$ are points in a circle of radius r, then the function $F(z)=A_0+A_1\sum_{j=1}^n m_j/(z-z_j)$, where A_0 and A_1 are arbitrary complex numbers, has at least $n-1$ zeros in the concentric circle of radius $2^{\frac{1}{2}}r$. In the present paper this theorem is extended to arbitrary positive m_j by use of the lemma that the equilateral hyperbola having any pair of zeros of $F(z)$ as vertices either passes through all the z_j or separates them. *M. Marden* (Milwaukee, Wis.).

v. Szökefalvi Nagy, Gyula. Die Nullstellen des Büschels von Polynomen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 786–808 (1942). (Hungarian. German summary)

Let $f(z)=(z-a_1) \cdots (z-a_n)$, $g(z)=(z-b_1) \cdots (z-b_n)$, and $h(z)=f(z)-Ag(z)$. By study of the argument and modulus of $f(z)/g(z)$ the following theorems are proved. (I) Let $0<m\leqq\arg A\leqq\pi$ and let $m_1+\cdots+m_n=m$ with all m_j positive. Then every zero of $h(z)$ falls in at least one of the regions composed of the points from which the line-segment a_j, b_j subtends an angle of not less than m_j. (II) Let $|A|\leqq M$ and let $M_1 \cdots M_n=M$ with all M_j positive. Then every zero of $h(z)$ lies in at least one of the circles $|z-a_j|\leqq M_j|z-b_j|$. (III) If all the zeros of $f(z)$ and $g(z)$ lie in a circle of radius R and if $|A|\leqq N^n<1$, then all the zeros of $h(z)$ lie in the concentric circle of radius $R(1+N)/(1-N)$. *M. Marden* (Milwaukee, Wis.).

v. Szökefalvi Nagy, Gyula. Der Wertvorrat von Polynomen in gewissen Bereichen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 1–12 (1943). (Hungarian. German summary)

The author proves principally the following two theorems. (I) Let $f(z)=a_0(z-z_1) \cdots (z-z_n)$ and let a and B be complex numbers such that $0<\arg B/f(a)=A\leqq\pi$. Let $A_j>0$ and $A_1+\cdots+A_n=A$. Let W_j be the angle which has its vertex at z_j and an aperture of $2A_j$ and is bisected by the line from z_j to point a. Then $f(b)\neq B$ at any point b common to all the W_j. (II) Let $u_j>0$ for all j and $u_1 \cdots u_n=|B/f(a)|$. Let K_j be the circle $|z-z_j|=u_j|a-z_j|$. Then $f(b)\neq B$ at any point inside or outside all the K_j. [Reviewer's note: these theorems follow at once from consideration of the argument and modulus of $f(b)/f(a)$.] *M. Marden.*

Lipka, Stephan. Über den Satz von Newton-Sylvester. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 80–87 (1943). (Hungarian. German summary)

Let $f(x)$ be a polynomial with real coefficients; $f^{(\nu)}(x)$ denotes the νth derivative of $f(x)$;

$$f_\nu(x)=\frac{(n-\nu)!}{n!}f^{(\nu)}(x); \quad F_\nu(x)=f_\nu^2(x)-f_{\nu-1}(x)f_{\nu+1}(x),$$
$$\nu=0, 1, \cdots, n.$$

The author proves several results about the real roots of $f(x)$, e.g., $f(x)$ has only real roots, if and only if $F_\nu(x)\geqq 0$ for all ν and all real x. Another result is that, if all the coefficients of all the $F_\nu(x)$ are positive, then the number of changes of sign of the coefficients of $f(x)$ equals the number of positive roots of $f(x)$. *P. Erdös* (Syracuse, N. Y.).

Erdös, P. Some remarks on polynomials. Bull. Amer. Math. Soc. **53**, 1169–1176 (1947).

The author proves a number of disconnected results, among which are the following. (1) If $a_0z^n+a_1z^{n-1}+\cdots+a_n$ ($a_0\neq 0$) has integer coefficients and its roots either all have absolute value 1 and are different or all satisfy $|z_i|<1$, then we have $|a_1|<n\epsilon$ for $n>n_0(a_0, \epsilon)$ [conjectured by I. Schur, Math. Z. 1, 377–402 (1918), p. 398].

(2) Let M be a point-set and let $\omega_n(M, z_0)$ denote the upper bound of $|f'(z_0)|$ for all polynomials $f(z)$ which satisfy $|f(z)|\leqq 1$ for $z\epsilon M$ [cf. Szegö, Math. Z. **23**, 45–61 (1925), and Fekete, Math. Z. **26**, 324–344 (1927)]. The author shows that $\omega_n(M, z_0)^{1/n}$ need not converge for $n\to\infty$ if $z_0\epsilon M$, and may be bounded if $z_0\epsilon M$ and the transfinite

diameter of M is 0. These questions were left open by Szegö and Fekete.

(3) If the polynomial $f(z)$ of degree n satisfies $-1\leqq f(x)\leqq 1$ for $-1\leqq x\leqq 1$, then for $|z|\geqq 1$ (z real or complex) we have $|f(z)|\leqq|T_n(z)|$; T_n is Chebyshev's polynomial.

N. G. de Bruijn (Delft).

Nikol'skiĭ, S. M. On the best linear method of approximation in the mean to differentiable functions by polynomials. Doklady Akad. Nauk SSSR (N.S.) **58**, 185–188 (1947). (Russian)

Let $W^{(s-1)}$ be the class of functions $f(x)$, $-1\leqq x\leqq 1$, having the $(s-1)$th derivative $\varphi(x)$ of bounded variation and such that the total variation of φ is at most 1. Let $E_n[f]_L$ be the minimum of $\int_{-1}^1|f-P_n|dx$ for all polynomials $P_n=a_0+a_1x+\cdots+a_nx^n$ of degree n, and let $\mathcal{E}_n[W^{(s-1)}]_L=\sup E_n[f]_L$ for all $f\epsilon W^{(s-1)}$. The author shows that $\mathcal{E}_n[W^{(s-1)}]_L$ remains unchanged if we require that the $(s-1)$th derivative φ of f is absolutely continuous and of total variation at most 1. Furthermore, he shows that $\mathcal{E}_n[W^{(s-1)}]$ is either

$$[2(s-1)!]^{-1}\max_{-1\leqq a\leqq 1}E_n(|a-x|^{s-1})_L \simeq[2(s-1)!]^{-1}E_n[|x|^{s-1}]_L \simeq 4\pi^{-1}n^{-s}\sum_{\nu=0}^{\infty}(-1)^\nu(2\nu+1)^{-s-1},$$

or

$$[2(s-1)!]^{-1}\max_{-1\leqq a\leqq 1}E_n[|a-x|^{s-1}\operatorname{sgn}(a-x)]_L \simeq[2(s-1)!]^{-1}E_n[|x|^{s-1}\operatorname{sgn} x]_L \simeq 4\pi^{-1}n^{-s}\sum_{\nu=0}^{\infty}(2\nu+1)^{-s-1}$$

according as s is even or odd ($n\to\infty$). Let

$$P^{(s)}(x,a)=\sum_{k=0}^{n}\alpha_k^{(s)}(a)x^k$$

be the polynomial of degree n giving the best approximation (in the L-metric) over $(-1,1)$ to $|a-x|^{s-1}$ when s is even and to $|a-x|^{s-1}\operatorname{sgn}(a-x)$ when s is odd. Then the linear method

$$U_n(f,x)=[2(s-1)!]^{-1}\sum_{k=0}^{n}x^k\int_{-1}^{1}\alpha_k^{(s)}(t)d\varphi(t),\qquad \varphi=f^{(s-1)},$$

of approximation of f (in L) by polynomials of degree n is the best method for the class $W^{(s-1)}$.

A. Zygmund.

Cenov, I. V. Certain questions of the theory of approximation of functions. Mat. Sbornik N.S. **21**(63), 435–438 (1947). (Russian)

Let $f(x)$ and $\varphi(x)$ be differentiable $n+1$ times in an interval $[a,b]$. Let $x_0, x_1, \cdots, x_n$ be distinct points of $[a,b]$ and let $f_n(x)$, $\varphi_n(x)$ be the Lagrange interpolating polynomials of f, φ corresponding to the fundamental points x_i. Finally, let $R_{n+1}(f)=f-f_n$, $R_{n+1}(\varphi)=\varphi-\varphi_n$. Using the fact that the function $\Phi(z)$ equal to

$$[f(x)-f_n(x)][\varphi(z)-\varphi_n(z)]-[f(z)-f_n(z)][\varphi(x)-\varphi_n(x)]$$

vanishes at $n+2$ points $x, x_0, \cdots, x_n$, we get the formula

$$R_{n+1}(f)\varphi^{(n+1)}(\xi)=R_{n+1}(\varphi)f^{(n+1)}(\xi).$$

This leads to the following consequences. (1) If

$$(*)\qquad |f^{(n+1)}(x)|<|\varphi^{(n+1)}(x)|,\qquad a\leqq x\leqq b,$$

then $|R_{n+1}(f)|<|R_{n+1}(\varphi)|$ for every system $x_0,\cdots,x_n$ in $[a,b]$. (2) Under the assumption (*),

$$\max_{a\leqq x\leqq b}|R_{n+1}(f)|<\max_{a\leqq x\leqq b}|R_{n+1}(\varphi)|.$$

(3) If (*) holds, then $E_n(f)<E_n(\varphi)$, where $E_n(f)$ is the best approximation of f by polynomials of degree not exceeding n. (4) The polynomial of degree not exceeding n giving the minimum of $\int_a^b(f-P)^2dx$ is a Lagrange interpolating polynomial for f. (5) If (*), then $\int_a^bR_{n+1}^2(f)dx<\int_a^bR_{n+1}^2(\varphi)dx$. Similar results are obtained for interpolating polynomials involving higher derivatives, and in the case when (*) is replaced by the inequality $|\Delta_s^{n+1}(f)|<|\Delta_s^{(n+1)}(f)|$ for $(n+1)$th differences.

A. Zygmund (Chicago, Ill.).

Huff, William N. The type of the polynomials generated by $f(xt)\varphi(t)$. Duke Math. J. **14**, 1091–1104 (1947).

Polynomial sequences $\{y_n(x)\}$ defined by a generating function of the form $f(xt)\phi(t)=\sum_0^\infty y_n(x)t^n$, with $f(x)=\sum_0^\infty a_nx^n/n!$, $\phi(t)=\sum_0^\infty b_nt^n/n!$, $b_0\neq 0$, $a_n\neq 0$ ($n=0,1,\cdots$), were considered by W. C. Brenke [Amer. Math. Monthly **52**, 297–301 (1945); these Rev. **7**, 64]. The present paper considers two problems concerning such sequences $\{y_n(x)\}$. (I) It determines necessary and sufficient conditions that $\{y_n(x)\}$ be respectively of (finite) A-type, B-type, C-type, as defined by the reviewer [Duke Math. J. **5**, 590–622 (1939); these Rev. **1**, 15]. (II) Conditions are obtained in order that a sequence $\{y_n(x)\}$ (as defined above) satisfy a linear differential equation with polynomial coefficients. Thus, a necessary and sufficient condition that $\{y_n(x)\}$, of finite A-type k, satisfy a differential equation of the form

$$\sum_{r=1}^{s}A_r(x)y_n^{(r)}(x)=[nl_{11}+n(n-1)l_{22}+\cdots +n(n-1)\cdots(n-p+1)l_{pp}]y_n(x),$$

where $s=(k+1)pl$ and $A_r(x)$ is a polynomial of degree at most r, is that $\phi(t)=e^{H(t)}$, $H(t)$ being a polynomial of degree l. The author also briefly considers recurrence relations and orthogonality.

I. M. Sheffer (State College, Pa.).

Special Functions

Buchholz, Herbert. Bemerkungen zu einer Entwicklungsformel aus der Theorie der Zylinderfunktionen. Z. Angew. Math. Mech. **25/27**, 245–252 (1947). (German. Russian summary)

The author proves the so-called Kneser-Sommerfeld expansion and three new related partial fraction expansions of combinations of Bessel functions. All four expansions are useful in the problem of electromagnetic wave-propagation in hollow cylinders. He notes that the Kneser-Sommerfeld expansion was given correctly by Sommerfeld and Kneser, but is incorrectly recorded in Watson's Treatise on the Theory of Bessel Functions [Cambridge University Press, 1922, p. 499] and in Magnus and Oberhettinger's Formeln und Sätze für die speziellen Funktionen der mathematischen Physik [Springer, Berlin, 1943, p. 26; these Rev. **9**, 183]. The present proof proceeds along the lines of that given in Watson's book. The paper contains also the sums of several infinite series involving zeros of the derivative of Bessel functions of the first kind. (The corresponding results for the zeros of the Bessel functions themselves were already known.)

A. Erdélyi (Pasadena, Calif.).

Kuznecov, P. I. Lommel functions of two imaginary arguments. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 555–560 (1947). (Russian)

The author derives recurrent and differential relations for the functions $U_n=\sum_{m=0}^{\infty}(-)^m(w/z)^{\pm(n+2m)}J_{n+2m}(z)$, where J is the Bessel function. A table of U_1 and U_2 with six decimals is given for $0\leqq w, z\leqq 6$, in steps of $w, z=1$. These functions occur in laminar motions of viscous fluids [W. Müller, Z. Angew. Math. Mech. **13**, 395–408 (1933)]. For other applications cf. the author's paper [same vol., 267–270 (1947); these Rev. **9**, 30]. *I. Opatowski.*

Bouwkamp, C. J. On spheroidal wave functions of order zero. J. Math. Phys. Mass. Inst. Tech. **26**, 79–92 (1947).

The present paper is based on the author's thesis [Groningen, 1941; these Rev. **8**, 179] and contains also some additional new information. Starting from the differential equation of spheroidal wave functions of order zero the eigenvalues are obtained from a continued fraction formula, the solution of which is only possible by approximation. The author discusses his method of improving approximate values such that two-decimal accuracy is improved to six-decimal. He then applies his method to numerical calculations in the case of small values of the parameter in the differential equation. A power series for the eigenvalues as dependent on the parameter is given up to the eighth power. Then a table of spheroidal wave functions of order zero is given, subdivided into 9 separate tables. In a note added to the paper Blanch states that her previous paper [same J. **25**, 1–20 (1946); these Rev. **8**, 53] dealing with the improvement of approximate eigenvalues of Mathieu functions shows a number of essential differences from the above paper dealing with a different subject. Hence her results are not anticipated by Bouwkamp's thesis.

M. J. O. Strutt (Eindhoven).

Infeld, L. Recurrence formulas for Coulomb wave function. Physical Rev. (2) **72**, 1125 (1947).

[The author's initial was given incorrectly as I. in the original.] The author observes that the recurrence formulas for the Coulomb wave functions, derived by L. Powell [same Rev. (2) **72**, 626–627 (1947); these Rev. **9**, 184] are particular cases of the factorization method [same Rev. (2) **59**, 737–747 (1941); these Rev. **2**, 364]. For their derivation the explicit form of the solution is unnecessary and the recurrence formulas follow at once from the structure of the differential equation itself. *S. C. van Veen* (Delft).

Poli, L. Sur les équations intégrales dont le noyau est une fonction K_n de Bessel. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 191–198 (1947).

If $f(p)=p\int_0^\infty e^{-ps}h(s)ds$ is the operational image of $h(t)$ and $h(1/p)$ is the operational image of $\sigma(t)$ then

$$f(p)=2p\int_0^\infty K_0(2(ps)^{\frac{1}{2}})\sigma(s)ds. \tag{1}$$

The author uses this relationship to evaluate some integrals of the form (1) and also to solve the integral equation (1) for certain given $f(p)$. His examples involve Bessel and related functions and confluent hypergeometric functions. [The general result is not new; cf. N. A. Shastri, Proc. Indian Acad. Sci. Sect. A. **20**, 211–223 (1944); these Rev. **6**, 269.] *A. Erdélyi* (Pasadena, Calif.).

Poli, Louis. Le calcul symbolique à deux variables. Revue Sci. **85**, 616–617 (1947).

Proof of a rule for the operational calculus in two variables [the result is not new, cf. Erdélyi, Monatsh. Math. Phys. **46**, 132–156 (1937), p. 137]. This rule is used in a formal derivation of the relation

$$\sum_{n=1}^{\infty}L_n(x,y)/n=-\gamma-\log\{\max(x,y)\},$$

where γ is Euler's number and L_n is P. Humbert's generalisation to two variables of the Laguerre polynomial.

A. Erdélyi (Pasadena, Calif.).

Cansado Maceda, E. Characteristic functions of the Pearson distributions. I. Revista Mat. Hisp.-Amer. (4) **7**, 117–127 (1947). (Spanish)

The characteristic functions of Pearson distributions can be expressed in terms of confluent hypergeometric functions. The author lists the formulae in detail. He starts with the type I distribution and expresses the characteristic function in terms of Whittaker's confluent hypergeometric function $M_{k,m}(z)$. By specialising the parameters, and by constructing limiting cases, he obtains the characteristic functions of "transitional types," viz., type II (when the characteristic function can be expressed in terms of Bessel functions), the normal law, rectangular distribution, type III (in the last three cases the characteristic function reduces to elementary functions), types VIII, IX, XII. The other types will be considered in the second part of the article.

A. Erdélyi (Pasadena, Calif.).

Cansado Maceda, E. On the factorial characteristic function. Revista Mat. Hisp.-Amer. (4) **7**, 159–164 (1947). (Spanish)

The factorial characteristic function $\omega(z)$ of the distribution function $F(x)$ is defined as $\omega(z)=\int_{-\infty}^{\infty}(1+z)^xdF(x)$. The expansion of $\omega(z)$ in powers of z generates the factorial moments. If the stochastic variable X assumes only the values $x=0, 1, 2, \cdots$ with probabilities p_x, then $\omega(z)$ is a power series in $1+z$, convergent at least in the circle $|z+1|\leqq 1$. The probabilities p_x can be expressed in terms of the derivatives of $\omega(z)$ at $z=-1$, and the factorial moments in terms of the derivatives of $\omega(z)$ at $z=0$. The results are applied to the binomial distribution, the Poisson distribution and Pólya's distribution. *A. Erdélyi.*

Shanker, Hari. On integral representations for the product of two Whittaker functions. J. London Math. Soc. **22**, 112–115 (1947).

The author obtains, by operational methods, an integral representation for $W_{k,m}(iz)W_{k,m}(-iz)$: in the integrand he has a product of a modified Bessel function of the third kind with a generalised hypergeometric function. [The reader should note that the symbol k instead of the accepted symbol K is used to denote modified Bessel functions of the third kind.] By specialising the parameters, several more integral representations are obtained. There is also a derivation by operational methods of a known integral representation. *A. Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

***Evans, Griffith C. Kellogg's uniqueness theorem and applications.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 95–104. Interscience Publishers, Inc., New York, 1948. $5.50.

In a space of a finite number of leaves (analogous to a Riemann surface) let T be a bounded domain with boundary consisting of a finite number of branch curves $s_1, \cdots, s_n$ and a bounded frontier T^*. It is assumed that topologically T is equivalent to an m-leaved sphere and the s_i to branch circles within the sphere; no two of the s_i should loop or have points in common. Given any point set E of T the base set $\bar{E}$ is defined as the set of points in ordinary space whose coordinates are the same as those of the points of E. Limit points are defined in such a way that if E is an infinite set on $T+T^*+\sum s_i$ it has a limit point on $T+T^*+\sum s_i$. The following generalization of Kellogg's uniqueness theorem is given. Let $w(M)$ be subharmonic in T and have the finite least upper bound m in T. Take $\epsilon>0$ and let e be the set of points Q on $T^*+\sum s_i$ where $\lim\sup_{M\to Q} w(M)\geqq m-\epsilon$, M in T. Then the base set $\bar{e}$ of e is closed and of positive capacity. [See O. D. Kellogg, Foundations of Potential Theory, Springer, Berlin, 1929, p. 335.] The author also generalizes Green's theorem. Let T^* consist of a finite number of regular surface elements, with exterior normal ν. Let u and v be functions harmonic and bounded in T and continuous together with their first partial derivatives on T^*. Then

$$\int_{T^*}\left(u\frac{dv}{d\nu}-v\frac{du}{d\nu}\right)dP=0.$$

The paper concludes with an application to univalent harmonic functions. *F. W. Perkins* (Hanover, N. H.).

Brelot, Marcel. Quelques applications de la topologie de R.-S. Martin dans la théorie des fonctions harmoniques. C. R. Acad. Sci. Paris **226**, 49–51 (1948).

L'auteur annonce des résultats qui prolongent ceux de R. S. Martin [Trans. Amer. Math. Soc. **49**, 137–172 (1941); ces Rev. **2**, 292] relatifs aux fonctions harmoniques "minimales" dans un domaine Ω. Brelot prend pour Ω un domaine (ouvert) de l'espace euclidien (à dimension $n\geqq 2$) compactifié par un point ∞ [Brelot, Ann. Sci. École Norm. Sup. (3) **61**, 301–332 (1944); ces Rev. **7**, 204], et suppose que le complémentaire de Ω est non polaire. Martin a défini une frontière idéale Δ telle que $\Omega\cup\Delta$ soit compact; un point $P\epsilon\Omega$ tend vers un point $M\epsilon\Delta$ si et seulement si, pour chaque couple $P_1\epsilon\Omega$, $P_2\epsilon\Omega$, $G(P,P_1)/G(P,P_2)$ a une limite finie non nulle (G désigne la fonction de Green de Ω). On choisit $P_0\epsilon\Omega$, et on pose, pour $P\epsilon\Omega$, $M\epsilon\Delta$,

$$(1)\qquad K(M,P)=\lim_{Q\epsilon\Omega,\,Q\to M} G(Q,P)/G(Q,P_0).$$

Donc $K(M,P)$ est continue par rapport à l'ensemble des variables M et P; comme fonction de P, elle est harmonique positive dans Ω. La frontière idéale Δ admet une partition en 2 ensembles Δ' et Δ''; par définition, M appartient à Δ' (et est dit "minimal") si la fonction (de P) $K(M,P)$ est minimale, c'est-à-dire si toute $u(P)$, harmonique dans Ω, telle que $0\leqq u(P)\leqq K(M,P)$, a la forme $cK(M,P)$ (c constante). D'après Martin, toute fonction harmonique $u(P)>0$ dans Ω définit une mesure de Radon positive μ_u et une seule qui soit portée par Δ' et satisfasse à $u(P)=\int K(M,P)d\mu_u(M)$.

Brelot définit la notion de fonction harmonique u "associée à 0" en un point $M_0\epsilon\Delta$; cela exprime l'existence d'un voisinage ouvert δ de M_0, dans $\Omega\cup\Delta$, tel que la solution du problème de Dirichlet-Wiener, pour $\delta\cap\Omega$, et pour une donnée frontière égale à u dans Ω et à 0 ailleurs, soit la fonction u elle-même. Pour que u harmonique positive soit associée à 0 en M_0, il faut et il suffit que M_0 n'appartienne pas au noyau fermé de la mesure correspondante μ_u; d'où une caractérisation des fonctions harmoniques minimales (i.e., harmoniques positives, associées à 0 en tout point de Δ sauf un).

La relation (1) entraîne que le potentiel $V(P)$ (par rapport à la fonction de Green de Ω) d'une mesure de Radon ν portée par un compact $\Sigma\subset\Omega$ a une "dérivée normale" en tout point $M\epsilon\Delta$, dans le sens suivant: $\lim_{P\to M} V(P)/G(P_0,P)$ existe; c'est une fonction continue $V_1(M)$ appelée "pente" de V en M. Dans $\Omega-\Sigma$, $V(P)$ est harmonique, bornée au voisinage de Δ, et s'annule en tout point frontière régulier de Ω; la réciproque est exacte; une telle fonction harmonique dans $\Omega-\Sigma$ admet donc une "pente" $V_1(M)$ en tout point $M\epsilon\Delta$ (en outre, si la fonction $V(P)$ est positive dans $\Omega-\Sigma$, $V_1(M)$ est positive en tout point $M\epsilon\Delta$). Si $V_1(M)=0$ pour tout $M\epsilon\Delta'$, $V(P)$ est identiquement nulle au voisinage de Δ; mais il est possible que $V_1(M)$ soit partout très petite sur Δ, sans que $V(P_0)$ soit très petite (P_0 point donné à l'avance): phénomène d'instabilité. *H. Cartan* (Paris).

Ertel, Hans. Über die Unstetigkeiten der zweiten Ableitungen des Schwerepotentials an Diskontinuitätsflächen der Dichte. Z. Angew. Math. Mech. **25/27**, 186–189 (1947). (German. Russian summary)

Let x, y, z be Cartesian coordinates with respect to axes fixed relatively to the rotating earth, the z-axis coinciding with the axis of rotation. Let $F(x,y,z)=0$ be the equation of a surface of discontinuity of the density ρ of the earth. The author represents by $\{\rho\}$ the jump of ρ across $F=0$, and writes the potential in the form

$$\Phi(x,y,z)=-\kappa\int\int\int\rho r^{-1}d\tau-\tfrac{1}{2}(x^2+y^2)\cdot\omega^2,$$

where κ is the gravitational constant and ω is the angular velocity of the earth's rotation. For suitably restricted F he derives formulas for the jumps in the second partial derivatives of Φ in terms of $\{\rho\}$ and the direction cosines of the normal n to $F=0$. He then proves that, if r and s are any two directions, not necessarily orthogonal, the jump in $\partial^2\Phi/\partial r\partial s$ is given by the formula

$$\left\{\frac{\partial^2\Phi}{\partial r\partial s}\right\}=\left\{\frac{\partial^2\Phi}{\partial s\partial r}\right\}=4\pi\kappa\{\rho\}\cos(n,r)\cos(n,s).$$

It is also shown that the jump in the mean curvature of an equipotential surface with normal N at intersection with $F=0$ is $\{K\}=4\pi\kappa g^{-1}\{\rho\}\sin^2(n,N)$. *F. W. Perkins.*

Bouwkamp, C. J. A new method for computing the energy of interaction between two spheres under a general law of force. Physica **13**, 501–507 (1947).

The author points out that, if a field of force has the property that the mutual energy V of two spheres is the same as that of two appropriate masses concentrated at the centers O, O' of those spheres, then $V=f(a,b)g(OO')$, where f depends only on the radii a, b of the two spheres and g is a function of the distance OO' only. If the potential between two point masses is of either of the following types: (i) $r^{-1}\exp(-kr)$, (ii) $r^{-1}\sin(kr)$, (iii) $r^{-1}\cos(kr)$, r being the mutual distance of the two masses, the energy of two

spheres is of the above type. The explicit expression of V in these three cases is essentially a product (P) of an exponential function by two Bessel functions of order 3/2. The author remarks that, if the potential between two point masses is a sum or an integral with respect to k of a product of a function of k by (i), (ii) or (iii), the corresponding mutual energy of two spheres is obtained from (P) by a similar operation of summation or integration. That energy is then calculated for a generalized Newtonian field (force proportional to r^n) as an application of simple formulae of the Laplace transformation. *I. Opatowski.*

Differential Equations

Buchheim, W. Herleitung der Lösungen linearer homogener Differentialgleichungen mit konstanten Koeffizienten im Falle mehrfacher Wurzeln der charakteristischen Gleichung mittels der Operatorenrechnung. Z. Angew. Math. Mech. **25/27**, 63 (1947).

Duffin, R. J. Nonlinear networks. IIa. Bull. Amer. Math. Soc. **53**, 963–971 (1947).

[For part I see the same Bull. **52**, 833–838 (1946); these Rev. **8**, 244.] The author considers the steady state behavior of a network which is defined as a collection of wires and batteries interconnected. If each conductor is such that the current through it is a nondecreasing function of the voltage across it and conversely then it is called quasi-linear. The author proves several theorems including one that "a network of quasi-linear conductors has a stable state of currents and this state is unique." The author also considers an analogue to the electrical problem which he calls an elastic network. It consists of a collection of springs joined at junction points and with forces applied at junction points. Here a vector formulation is given. *N. Levinson.*

Rocard, Yves. Étude de la stabilité des systèmes accessibles à des mesures. Revue Sci. **85**, 519–531 (1947).

The author discusses the Nyquist and similar graphical criteria for stability and illustrates their use by examples involving various electric circuits and time-lag control systems. *M. Marden* (Milwaukee, Wis.).

Sansone, Giovanni. Studi asintotici sulle equazioni differenziali di secondo ordine. Rend. Sem. Mat. Fis. Milano **15**, 115–128 (1941).

The author presents a survey of the theory of the asymptotic behavior of solutions of linear equations of the form $y''+A(t)y=0$. *R. Bellman* (Princeton, N. J.).

Wintner, Aurel. Stability and high frequency. J. Appl. Phys. **18**, 941–942 (1947).

The author shows by an example that $\omega(t)\rightarrow\infty$ is not a sufficient condition to assure the uniform boundedness of a solution of $x''+\omega^2x=0$ as $t\rightarrow\infty$. However if $d\omega(t)\geqq 0$ then this is the case. *N. Levinson* (Cambridge, Mass.).

Wintner, Aurel. Vortices and nodes. Amer. J. Math. **69**, 815–824 (1947).

The relationship between a pair of real differential equations (1) $x'=ax+by+f(x, y)$, $y'=cx+dy+g(x, y)$, and the trivial linear case (2) $x'=ax+by$, $y'=cx+dy$, where a, b, c and d are constants, is considered. The point (0, 0) is called an attractor if any solution $[x(t), y(t)]$ near enough to (0, 0) tends to (0, 0) as $t\rightarrow\infty$ (or as $t\rightarrow-\infty$). If $r^2=x^2+y^2$ and if f and g are $o(r)$ as $r\rightarrow 0$ then the author shows that (0, 0) is an attractor for (1) if it is for (2). If f and g are $O(r^{1+\epsilon})$, $\epsilon\rightarrow 0$ as $r\rightarrow 0$, then if (0, 0) is a vortex (sometimes called focal point) of (2) it is also a vortex point of (1) and a correspondence exists between solutions of (1) and (2). A similar situation prevails in case (0, 0) is what the author terms a proper node of (2). The condition $O(r^{1+\epsilon})$ can be weakened, as is indicated in the paper. *N. Levinson.*

Lang, H. A. Note on Rayleigh's method and the nonuniform strut. Quart. Appl. Math. **5**, 510–511 (1948).

The lowest eigenvalue P^* of $B(x)u''+Pu=0$, with $y(0)=y(L)=0$, can be approximated by

$$P=\int_0^L (y')^2dx \Big/ \int_0^L (y^2/B)dx,$$

when y is an approximation to the eigenfunction u. The author proves that this formula gives a lower bound for P^*, in contrast with the Rayleigh result which is an upper bound. *G. F. Carrier* (Providence, R. I.).

Faedo, Sandro. Il teorema di Fuchs per le equazioni differenziali lineari a coefficienti non analitici e proprietà asintotiche delle soluzioni. Ann. Mat. Pura Appl. (4) **25**, 111–133 (1946).

For the equation $y^{(n)}+p_{n-1}y^{(n-1)}+\cdots+p_0y=0$, in which p_i has the form $\mu_ix^{i-n}+q_i$ with constant μ's and, N being the maximum multiplicity of a root of the indicial equation, the integrals

$$\int^b x^{n-i-1}|q_i|\cdot|\log x|^{N-1}dx$$

are finite, the paper proves the existence of a fundamental set of solutions composed of subsets

$$x^k(1+u_0),\ x^k(\log x)(1+u_1),\ \cdots,\ x^k(\log x)^{\nu-1}(1+u_{\nu-1}),$$

where k is a root of the indicial equation with multiplicity ν. The u's are series converging absolutely and uniformly and differentiable termwise $n-1$ times in an interval $(0, c)$. The terms of each u are found recursively by integration, are continuous in $(0, c)$ and vanish for $x=0$. The result is a generalization of Fuchs's work to nonanalytic functions. It is noteworthy that there is no logarithmic factor when the roots are distinct, even if two differ by a positive integer. Under more restrictive assumptions previous extensions had been made for $n=2$ by M. Bôcher [Trans. Amer. Math. Soc. **1**, 40–52 (1900)] and U. Dini [Ann. Mat. Pura Appl. (3) **11**, 285–335 (1905)]. *J. M. Thomas* (Durham, N. C.).

Faedo, S. Sulla stabilità delle soluzioni delle equazioni differenziali lineari. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 564–570 (1947).

Faedo, S. Sulla stabilità delle soluzioni delle equazioni differenziali lineari. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 757–764 (1947).

Faedo, S. Sulla stabilità delle soluzioni delle equazioni differenziali lineari. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 37–43 (1947).

The stability of the solutions of

$$y^{(n)}+[a_1+\phi_1(x)]y^{(n-1)}+\cdots+[a_n+\phi_n(x)]y=b+\phi(x),$$

where a_i and b are constants and $\phi_i(x)$ and $\phi(x)$ are absolutely integrable over (x_0, ∞), is considered. Criteria are given when the roots of $z^n+a_1z^{n-1}+\cdots+a_n=0$ are all dis-

tinct as well as when multiple roots occur. The asymptotic behavior of the solutions of the homogeneous equation is first given and using the "variation of constants" formula results are obtained for the nonhomogeneous case. [The asymptotic behavior for a homogeneous linear system with the above hypotheses is given by O. Dunkel, Proc. Amer. Acad. Arts Sci. 38, 339–370 (1902). An exponential change of the independent variable puts Dunkel's system into a form corresponding to that given above.] *N. Levinson.*

Mitrinovitch, Dragoslav S. Sur une classe d'équations différentielles d'ordre supérieur. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 33, 521–526 (1947).

Let m, n, s $(m>n)$ be three positive integers and use the differential form

$$(1)\qquad \lambda = y^{(n)} + \sum_{i=1}^{n-1} b_i(x) y^{(n-i)}, \qquad y^{(k)} \equiv d^k y/dx^k,$$

to form the differential equation of order $H = \max(m, n+s)$:

$$(2)\qquad y^{(m)} + \sum_{k=1}^{m-1} a_k(x) y^{(m-k)} = \Phi(x, \lambda, \lambda^{(1)}, \cdots, \lambda^{(s)}).$$

If one is free to select the n functions $a_{m-n+1}, \cdots, a_m$, the author shows they can be so chosen that the solution of (2) will be reduced to a solution of an equation of order $h = \max(m-n, s)$ in λ,

$$(3)\qquad F(x, \lambda, \cdots, \lambda^{(h)}) = 0,$$

followed by the solution of equation (1) when λ is replaced by an integral of (3). *F. G. Dressel* (Durham, N. C.).

Germay, R. H. J. The generalization of a theorem of Jacobi and the integration of Koenig's systems of partial differential equations of the first order. Actas Acad. Ci. Lima 10, 3–16 (1947). (Spanish)

L'auteur considère un système d'équations aux dérivées partielles avec p fonctions inconnues et deux groupes de, respectivement, r et m variables; on suppose qu'il se compose de rp équations résolues par rapport aux dérivées relatives au premier groupe de variables, pendant que les deuxièmes membres dépendent de toutes les variables, des fonctions inconnues et de leur dérivées par rapport au deuxième groupe de variables; on suppose que, sous cette forme, le système est complet et on cherche une solution satisfaisant une condition de Cauchy, consistant en ce que les fonctions inconnues doivent se réduire à des fonctions données du deuxième groupe de variables pour des valeurs initiales données aux variables du premier groupe. Le problème a été complètement étudié par J. Koenig [Math. Ann. 23, 520–528 (1884)] qui donne des conditions nécessaires et suffisantes pour que le système soit complet et sa réduction, par une transformation de Mayer, à l'intégration d'un système analogue, de seulement p équations résolues selon des dérivées par rapport à une seule variable, tandis que les autres $r-1$ variables apparaissent comme paramètres. L'auteur observe que le dernier pas de cette intégration peut s'effectuer par l'application d'une généralization d'un procédé de Jacobi qu'il a publiée autrefois [Revista Mat. Hisp.-Amer. (1) 6, 65–80 (1924)]. *B. Levi* (Rosario).

Drach, Jules. Sur les équations aux dérivées partielles du premier ordre dont les caractéristiques sont lignes asymptotiques des surfaces intégrales. C. R. Acad. Sci. Paris 225, 1221–1224 (1947).

If the characteristics of the partial differential equation $p+f(x, y, z, q)=0$ are asymptotic lines of the integral surfaces, then $f_x+(qf_z-f)f_z+f_qf_y=0$. The author shows how the integral surfaces and their asymptotic lines may be obtained for equations of this type. In case f is free of z his method gives the solution in parametric form. The method presented for the general case is not always applicable and yields solutions only in very special simple cases. *M. S. Knebelman* (Pullman, Wash.).

Risser, René. D'un certain mode de recherche des surfaces de probabilités. C. R. Acad. Sci. Paris 225, 1266–1268 (1947).

The system of partial differential equations

$$\partial z/\partial x = zf(x, y)\phi^{-1}(x, y), \quad \partial z/\partial y = zg(x, y)\psi^{-1}(x, y),$$

is said to occur in probability theory. Here f, g, ϕ, ψ are given functions satisfying $\partial(f\phi^{-1})/\partial y = \partial(g\psi^{-1})/\partial x$. From the equations, $z^{-1}\partial^2 z/\partial x\partial y = H(x, y)$ can be expressed in terms of f, g, ϕ, ψ. Some examples are mentioned, e.g., when f, ϕ, and ψ are constants. *W. Feller* (Ithaca, N. Y.).

Bergman, S., and Schiffer, M. On Green's and Neumann's functions in the theory of partial differential equations. Bull. Amer. Math. Soc. 53, 1141–1151 (1947).

Cet article complète un travail antérieur [Bergman et Schiffer, Duke Math. J. 14, 609–638 (1947); ces Rev. 9, 187] relatif à l'équation (1) $\Delta\varphi = P(x, y)\varphi$, $P>0$, dans un domaine plan ω de frontière assez régulière. Posons

$$D(\varphi, \psi) = \iint (\text{grad } \varphi \cdot \text{grad } \psi + P\varphi\psi) dxdy.$$

Considérons la classe Ω des fonctions φ de gradient continu et $D(\varphi, \varphi)$ fini, la sous-classe Γ de celles qui sont intégrales de (1) et la sous-classe Ω_0 de celles qui s'annulent à la frontière. On dit qu'une suite $\varphi_\nu \epsilon \Omega$ est orthonormalisée si $D(\varphi_\mu, \varphi_\nu)$ vaut 0 ou 1 suivant que $\mu \neq \nu$, $\mu = \nu$. Elle sera dite complète par rapport à l'une des classes si toute fonction de la classe peut être approchée arbitrairement par une combinaison linéaire de φ_ν, uniformément sur tout compact. On avait montré que si φ_ν est orthonormalisée complète par rapport à Γ, $\sum_1^\infty \varphi_\nu(M)\varphi_\nu(P)$ est une fonction $K(M, P)$ indépendante de φ_ν qui vaut $[N(M, P) - G(M, P)]/(2\pi)$, où N et G sont les fonctions de Neumann et Green relativement à ω et (1). Le but principal de cet article est de montrer que N et G s'obtiennent, à peu près comme K, à partir de suites orthonormalisées, complètes respectivement par rapport à Ω ou Ω_0. *M. Brelot* (Grenoble).

Picone, M. Sulla traduzione in equazione integrale lineare di prima specie dei problemi al contorno concernenti i sistemi di equazioni lineari a derivate parziali. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 485–492 (1947).

Cette note fait suite à une autre du même titre [même tome, 365–371 (1947); ces Rev. 9, 145]; on y définit, en termes très généraux, quatre types de problèmes dont la méthode de l'auteur permet de calculer numériquement la solution: problème de Cauchy, problème uniforme, problème mixte et problème de propagation. Voici l'énoncé du dernier que l'auteur considère ensuite particulièrement: on pense la frontière $\mathfrak{F}D$ d'un domaine D divisée en trois parties qu'on appelle $\mathfrak{F}_1D$, $\mathfrak{F}_2D$, $\mathfrak{F}_3D$ telles que, pour la détermination d'une solution du système différentiel, on puisse donner arbitrairement les valeurs que certaines formes linéaires des dérivées des différents ordres des fonctions inconnues prennent sur $\mathfrak{F}_1D$, et une partie seulement des valeurs des mêmes formes sur $\mathfrak{F}_2D$, tandis qu'aucune condition n'est donnée sur $\mathfrak{F}_3D$. Comme application des

concepts généraux exposés dans la note précédente, on propose trois méthodes pour calculer un développement de Fourier de la solution. On considère ensuite en particulier le cas où les coefficients des équations du problème sont des constantes; si dans les opérateurs E_{hk} [voir la note précédente] on substitue au lieu des symboles différentiels $\partial^{i_1+\cdots+i_r}/\partial x_1^{i_1}\cdots\partial x_r^{i_r}$, les monômes correspondents $\xi_1^{i_1}\cdots\xi_r^{i_r}$, on obtient des polynômes, qu'on prend comme éléments d'une matrice analogue à E; en prenant les mineurs de cette matrice, multipliés par $e^{-\Sigma \xi_i x_i}$, comme coordonnées des vecteurs V_l, on obtient enfin une représentation dans laquelle les fonctions inconnues se développent en séries de polynômes. *B. Levi* (Rosario).

Picone, M. Sulla traduzione in equazione integrale lineare di prima specie dei problemi al contorno concernenti i sistemi di equazioni lineari a derivate parziali. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 717–725 (1947).

[Voir l'analyse ci-dessus.] Dans cette troisième note on applique les derniers développements de l'antérieure au cas des équations de l'élasticité; l'auteur démontre que certaines combinaisons linéaires des dérivées des vecteurs à coordonnées polynômes obtenus avec ce procédé-là satisfont alors les équations avec deuxième membre nul. Il appelle ces expressions des solutions polynomiales des équations homogènes de l'élasticité; il y en a $3(2\nu+1)$ indépendentes pour chaque valeur du degré ν; et toute solution de ce système homogène s'exprime par une série de celles-ci. L'auteur termine en énonçant quelques thèmes de recherche. *B. Levi* (Rosario).

Šerman, D. I. On certain cases of the general problem of the theory of steady state vibrations. Doklady Akad. Nauk SSSR (N.S.) **56**, 567–570 (1947). (Russian)

Let S be a finite simply-connected plane domain bounded by a smooth rectifiable Jordan curve L. The author investigates solutions $u(x, y; \lambda)$ of the differential equation $\Delta u - \lambda^2 u = 0$ in S satisfying a boundary condition of the form

$$\sum_{k=0}^{m}\sum_{j=0}^{k} a_{kj}(s)\frac{\partial^k u}{\partial x^{k-j}\partial y^j} = f(s)$$

on L, where $f(s)$ and the $a_{kj}(s)$ are continuous on L and the $a_{mj}(s)$ satisfy a Lipschitz condition. An expression for the general solution is obtained and it is shown that the solution of the problem is equivalent to the solution of an integral equation. *E. F. Beckenbach* (Los Angeles, Calif.).

Gevrey, Maurice. Sur le cas irrégulier du problème de la dérivée oblique lorsque le nombre des variables est supérieur à deux. C. R. Acad. Sci. Paris **225**, 1251–1253 (1947).

Let R be a bounded n-dimensional open region with boundary S. The author sketches a method for solving equations elliptic in R when the boundary conditions involve an oblique derivative on S (that is, a derivative in a direction varying continuously on S but not identically in the direction of the conormal). *F. G. Dressel.*

Rubinštein, L. I. On the determination of the position of the boundary which separates two phases in the one-dimensional problem of Stephan. Doklady Akad. Nauk SSSR (N.S.) **58**, 217–220 (1947). (Russian)

This paper gives another method for the solution of the problem which was studied by the author in an earlier paper [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 37–54 (1947); these Rev. **8**, 516]. The problem is reduced to the solution of a system of integral equations which is solved by the method of successive substitutions. The uniqueness of the solution is established. The difference between the present method and the one given in the earlier paper consists in the avoidance here of the nonlinear transformation of the coordinates used there, and in the fact that by the previous method the temperature gradient had to be taken into consideration within the regions of each of the phases, while by the new method that can be avoided. *H. P. Thielman.*

Rubinštein, L. I. On the question of the process of propagation of freezing in frozen soil. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **11**, 489–496 (1947). (Russian)

The propagation of freezing is studied for an infinite half-space and Stephan's boundary conditions in the general case when the soil's moisture is inhomogeneous and therefore possesses many different freezing temperatures. *E. Kogbetliantz* (New York, N. Y.).

Karimov, Dž. H. On periodic solutions of nonlinear differential equations of parabolic type. Doklady Akad. Nauk SSSR (N.S.) **58**, 969–972 (1947). (Russian)

The author considers the nonlinear equation

$$z_t - a^2 z_{xx} = \phi(x, t) + \mu f(z, z_x),$$

subject to $z(0, t) = z(\pi, t) = 0$, $z(x, t) = z(x, t+1)$, $0 \leqq x \leqq \pi$, $0 \leqq t \leqq 1$. The method of successive approximations is used as follows: $\partial z_{n+1}/\partial t - a^2\partial^2 z_{n+1}/\partial x^2 = \phi(x, t) + \mu f(z_n, \partial z_n/\partial x)$, to obtain a uniformly convergent subsequence $\{z_{n_k}\}$, converging to a function W. However, it is not shown that W satisfies the partial differential equation, and thus the result claimed must be regarded as unproved. *R. Bellman.*

Cârstoiu, Ion. Sur le calcul symbolique à deux variables et ses applications. C. R. Acad. Sci. Paris **226**, 45–47 (1948).

The author summarises some well-known rules of the operational calculus in two variables, and uses them to find formally the known solution of the boundary value problem $u_t(x, t) = ku_{xx}(x, t)$, $u(x, 0) = 0$, $u(0, t) = U(t)$, $x, t > 0$. *A. Erdélyi* (Pasadena, Calif.).

Comenetz, G. Continuous heating of a hollow cylinder. Quart. Appl. Math. **5**, 503–510 (1948).

Two problems are treated in the determination of the temperatures in infinitely long hollow cylinders initially at temperature zero. In the first problem the outer surface is kept at temperature zero and the flux of heat through all points of the inner surface is the same quadratic function of the time. In the second the outer surface is insulated and the flux through the inner surface is a linear function of the time. In both cases the temperature function is first written as the sum of several infinite series some of which converge rapidly and the rest slowly. The author then determines closed forms for the slowly convergent series. The formulas are useful in particular for predicting temperatures in barrels of guns during rapid firing. *R. V. Churchill.*

Mindlin, Ya. A. A general representation of solutions of the wave equation. Doklady Akad. Nauk SSSR (N.S.) **58**, 17–20 (1947). (Russian)

The author shows that solutions of the equation $a^2\sum_{i=1}^{n+2}\partial^2 u/\partial x_i^2 = \partial^2 u/\partial t^2$ which are zero at infinity can be

put in the form

$$u=\sum_{s=0}^{\infty}\int_0^{\infty} Y_s(r\cosh\xi, t, \theta_1, \cdots, \theta_n, \varphi)\sinh^n\xi\cdot C_s^n(\cosh\xi)d\xi,$$

where Y_s is a product of a hyperspherical function of order s by a function of t and $z=r\cosh\xi$, satisfying a one-dimensional wave equation in (z, t); C_s^n is a polynomial of Jacobi-Gegenbauer and $(r, \theta_1, \cdots, \theta_n, \varphi)$ are the $(n+2)$-dimensional spherical coordinates. *I. Opatowski* (Ann Arbor, Mich.).

Erdélyi, A. On certain discontinuous wave functions. Proc. Edinburgh Math. Soc. (2) 8, 39–42 (1947).

Let $f(x, y)$ denote a function of class C^2, and let $\sigma^2=c^2t^2-z^2$. It was noted by H. Bateman that the expression $W(x, y, z, t)$ defined by

$$W=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x+\sigma\cos\lambda,\quad y+\sigma\sin\lambda)d\lambda, & \sigma^2\geqq 0,\\ 0, & \sigma^2<0,\end{cases}$$

represents a solution of the wave equation

$$(1)\qquad c^2(W_{xx}+W_{yy}+W_{zz})=W_{tt}.$$

The author proves that (1) remains valid if f is permitted to have "jump" discontinuities in its first and second derivatives along a curve $h(x, y)=0$ of class C^2. This is proved by verifying that the extra terms introduced by the discontinuities cancel out. *F. John* (New York, N. Y.).

✶Asgeirsson, Leifur. Über Mittelwertgleichungen, die mehreren partiellen Differentialgleichungen 2. Ordnung zugeordnet sind. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 7–20. Interscience Publishers, Inc., New York, 1948. $5.50.

Ist die Funktion $u(x_1, \cdots, x_\nu; y_1, \cdots, y_\nu)$ regulär im Bereich (B_l) $(x_1^2+\cdots+x_\nu^2)^{\frac12}+(y_1^2+\cdots+y_\nu^2)^{\frac12}\leqq l$ (l positiv konstant) und in B_l Lösung der ultrahyperbolischen Differentialgleichung $\sum_{i=1}^{\nu}(u_{x_ix_i}-u_{y_iy_i})=0$, so ist der Mittelwert von $u(x_1, \cdots, x_\nu; 0, \cdots, 0)$ auf (und im Innern) der Kugel $\sum_{i=1}^{\nu}x_i^2=l^2$ gleich dem Mittelwert von $u(0, \cdots, 0; y_1, \cdots, y_\nu)$ auf (und im Innern) der Kugel $\sum_{i=1}^{\nu}y_i^2=l^2$.

In dieser Weise hatte Verf. einer jeden ultrahyperbolischen Differentialgleichung einen "Mittelwertsatz" zugeordnet. Handelt es sich um lineare homogene Differentialgleichungen vom Typus

$$\sum_{h,k=1}^{m} a_{hk}(x)u_{x_hx_k}+2\sum_{j=1}^{m}b_j(x)u_{x_j}+c(x)u=0,$$

so gelingt die Zuordnung eines entsprechenden Mittelwertsatzes, wofern mit Hilfe einer "Belegungsfunktion" $\varphi(\xi_1, \cdots, \eta_\nu)$ und der Transformation $x_i=\chi_i(\xi_1, \cdots, \eta_\nu)$ der Funktion u eine Funktion $U=\varphi(\xi_1, \cdots, \eta_\nu)u$ zugeordnet werden kann, die wiederum Lösung einer ultrahyperbolischen Differentialgleichung ist.

Für konstante Koeffizienten a_{hk}, b_j, c lassen sich die Transformationsfunktionen χ_i linear wählen. Nach dieser Wahl ergibt sich für die Belegungsfunktion $\varphi=e^A\psi$ (A lineare Funktion der $\xi_1, \cdots, \eta_\nu$) und für ψ eine weitere lineare Differentialgleichung zweiter Ordnung $L\psi=0$. Der resultierende Mittelwertsatz dieses Verfahrens erscheint so als eine die (gleichberechtigten) Funktionen u und ψ verknüpfende Relation und damit einem System (mindestens zweier) linearer partieller Differentialgleichungen zweiter Ordnung zugeordnet. Diese von Verf. bereits früher [vgl. C. R. Congrès Internat. Math., Oslo, 1936, Bd. 2, S. 51–53] als "Kombinationsverfahren" eingeführte Methode wird in der vorliegenden Arbeit an den folgenden Beispielen eingehend durchgeführt:

$u_{xx}+u_{yy}+ku=0,\quad v_{xx}+v_{yy}+kv=0,\quad k=\text{konstant};$
$u_{xx}+u_{yy}-u_{ss}+ku=0,\quad v_{xx}+v_{yy}-v_{ss}+kv=0,\quad k=\text{konstant};$
$u_{xx}+u_{yy}-u_{ss}=0,\quad v_{xx}+v_{yy}=0;$
$u_{xx}+u_t=0.$ *M. Pinl* (Köln).

✶Hadamard, J. Sur le cas anormal du problème de Cauchy pour l'équation des ondes. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 161–165. Interscience Publishers, Inc., New York, 1948. $5.50.

Un problème d'une équation différentielle est appelé un problème correctement posé, si la solution existe, si elle est définie uniquement, et si elle dépend des conditions initiales d'une manière continue. Si ces conditions n'ont pas lieu il s'agit d'un problème anormal. Le problème de Cauchy des équations différentielles hyperboliques

$$(e_3)\qquad u_{xx}+u_{yy}+u_{ss}-u_{tt}=0,$$
$$(e_2)\qquad u_{xx}+u_{yy}-u_{tt}=0$$

des ondes sphériques (e_3) ou des ondes cylindriques (e_2) est posé correctement pour une variété orientée dans l'espace, tandis qu'il est anormal pour une variété orientée dans le temps. En cas anormal l'auteur formule le problème (A): les valeurs de la dérivée normale $\partial u/\partial x$ d'une solution u d'une des équations (e_2), (e_3) le long du plan $x=0$ étant supposées identiquement nulles, comment pourront être choisies les valeurs de la fonction u pour que le problème de Cauchy ainsi défini ait une solution? Ce problème peut être remplacé par celui où l'on suppose que u soit identiquement zéro sur $x=0$ en posant la question, comment les valeurs $u_1=(\partial u/\partial x)_{x=0}$ doivent être choisies pour donner lieu à un problème de Cauchy possible.

Pour déterminer la fonction u l'auteur applique sa théorie des moyennes hémisphériques (cas (e_3)) ou des moyennes semicirculaires (cas (e_2)) $W(y_0, z_0, t_0)$ ou $W(y_0, t_0)$ d'une fonction $U(x, y, z)$ ou $U(x, y)$ sur l'hémisphère de "centre" $x=0, y_0, z_0$ et de rayon t_0 ou sur la demi-circonférence de "centre" $x=0, y_0$ et de rayon t_0 et formule (par exemple, en cas (e_3)) le problème (B): la fonction $W(y_0, z_0, t_0)$ étant donnée, la fonction $U(x, y, z)$ existe-t-elle, et comment peut-on la calculer? Sur certaines conditions on peut calculer les fonctions $U(x, y, z)$ ou $U(x, y)$ par les valeurs moyennes $W(y_0, z_0, t_0)$ ou $W(y_0, t_0)$ en un point quelconque d'une telle hémisphère ou d'une telle demi-circonférence. Il en résulte qu'une telle fonction de valeur moyenne W est loin de pouvoir être choisie arbitrairement. Si la fonction W est connue dans un segment cylindrique d'une épaisseur arbitrairement petite, elle admet un prolongement analytique limité par des caractéristiques dans l'intérieur d'un double cône.

Les équations (e_3) et (e_2) sont invariantes vis à vis du groupe de Lorentz. L'auteur discute comment on peut désigner parmi les solutions du problème (B) celles qui sont invariantes de la même manière. *M. Pinl* (Cologne).

✶Lewy, Hans. On the convergence of solutions of difference equations. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 211–214. Interscience Publishers, Inc., New York, 1948. $5.50.

The convergence of the solutions of difference equations as the length of the elementary mesh size tends to zero has

been studied in extenso for the case of hyperbolic equations by R. Courant, K. Friedrichs and the author [Math. Ann. 100, 32–74 (1928)]. This paper illustrates the situation for the special example of the differential equation of cylindrical waves, $u_{zz}=u_{xx}+u_{yy}$, with the conditions $u=f(x, y)$, $u_z=0$ for $z=0$, where $f(x, y)$ is supposed to vanish of sufficiently high order as x or y tends to infinity.

Let $e^{i(\alpha x+\beta y+\gamma z)}$ be a solution of the partial difference equation associated with the problem. Then we get the following condition for the coefficients α, β, γ and the mesh sizes h, k, l in the (x, y, z)-directions:

$$l^{-2}\sin^2 \tfrac{1}{2}\chi l=h^{-2}\sin^2 \tfrac{1}{2}\alpha h+k^{-2}\sin^2 \tfrac{1}{2}\beta k, \quad l^{-2}\geqq h^{-2}+k^{-2},$$

for a real γ. For γ we get the Taylor series

$$\gamma=(\alpha^2+\beta^2)^{\frac{1}{2}}\left(1-\frac{\alpha^4h^2+\beta^4k^2-(\alpha^2+\beta^2)^2l^2}{6(\alpha^2+\beta^2)}+\cdots\right)\to(\alpha^2+\beta^2)^{\frac{1}{2}}, \qquad h, k, l\to 0.$$

Then let $2g$ be the Fourier transform of $f(x, y)$:

$$f(x, y)=\int_{-\infty}^{\infty}\int 2g(\alpha, \beta)e^{i(\alpha x+\beta y)}d\alpha d\beta.$$

Then we get

$$u=\int_{-\infty}^{\infty}\int g(\alpha, \beta)(e^{i(\alpha x+\beta y+\gamma z)}+e^{i(\alpha x+\beta y-\gamma z)})d\alpha d\beta$$

as the solution of the difference problem. The boundedness of $e^{i(\alpha x+\beta y+\gamma z)}$ and $e^{i(\alpha x+\beta y-\gamma z)}$ for all real α, β, x, y, z together with the continuity of γ as a function of α, β, h, k, l in any bounded region of the (α, β)-plane as h, k, l tend to zero imply the convergence of u, as h, k, l tend to zero, to the solution

$$\int_{-\infty}^{\infty}\int g(\alpha, \beta)(e^{i(\alpha x+\beta y+(\alpha^2+\beta^2)^{\frac{1}{2}}z)}+e^{i(\alpha x+\beta y-(\alpha^2+\beta^2)^{\frac{1}{2}}z)})d\alpha d\beta$$

of the differential equation of cylindrical waves. The condition $l^{-2}\geqq h^{-2}+k^{-2}$ admits a geometrical interpretation.

M. Pinl (Cologne).

Cinquini-Cibrario, M. Una proprietà delle superficie integrali delle equazioni non lineari di ordine n di tipo iperbolico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 49–55 (1947).

The author proves a generalization of the following theorem. Let Γ be a curve common to two integral surfaces of a nonlinear second order hyperbolic differential equation in two independent variables. If Γ is a characteristic curve for one of the surfaces and if the surfaces have contact of order 2 in one point of Γ, then the surfaces have contact of order 2 all along Γ and Γ is characteristic with respect to both surfaces.

The author's generalization applies to a hyperbolic equation of nth order for functions $z(x, y)$. Let Γ be a curve along which two integral surfaces have contact of order $n-2$. Let Γ be characteristic with respect to one of the surfaces and let the surfaces have contact of order n in one point of Γ. Then (subject to certain regularity conditions) the surfaces have contact of order n all along Γ and Γ is a characteristic curve with respect to both surfaces.

In case the hyperbolic differential equation is quasilinear, we get a simpler result. Let two integral surfaces have contact of order $n-2$ along a curve Γ, and contact of order $n-1$ in one point of Γ. Then, if Γ is characteristic with respect to one surface, the surfaces have contact of order $n-1$ all along Γ, and Γ is characteristic for both surfaces. *F. John* (New York, N. Y.).

Difference Equations

Strodt, Walter. Principal solutions of difference equations. Amer. J. Math. 69, 717–757 (1947).

In this paper the author gives a new definition of principal solutions of difference equations with analytic coefficients. The difference equation

$$(1) \qquad f[x, y(x+\omega_1), \cdots, y(x+\omega_n)]=0,$$

in which ω_k are given complex numbers, and $f(x, y_1, \cdots, y_n)$ is a polynomial in the y_k with coefficients analytic in a certain region, is approximated by an infinite sequence of q-difference equations each of which is of the form

$$(2) \qquad f[x, y\{T_1(x)\}, \cdots, y\{T_n(x)\}]=0,$$

where $T_k(b)=b$ for some complex number b, and $k=1, \cdots, n$. Special solutions of equation (2) are defined as solutions expansible in ascending, not necessarily integral, powers of x; a primary solution $y(x)$ of (1) is defined with respect to a certain region as a solution analytic in this region if suitably chosen special solutions of the approximating q-difference equations converge to $y(x)$ in this region. A principal solution $y(x)$ of (1) is defined with respect to a certain region if, for every closed bounded subset S of this region and every positive number ϵ, there exists an ordered set of complex numbers $z_1, \cdots, z_n$ such that $(\sum_{k=1}^{n}|z_k-1|^2)^{\frac{1}{2}}<\epsilon$, and such that for some primary solution $y(x, z_1, \cdots, z_n)$ of the equation

$$f[x, z_1y(x+\omega_1), \cdots, z_ny(x+\omega_n)]=0,$$

the inequality $|y(x, z_1, \cdots, z_n)-y(x)|<\epsilon$ is valid for every x in S. The author shows the existence of principal solutions for a broad class of linear and nonlinear equations, which include in particular most algebraic difference equations with coefficients analytic at ∞, and equations of the types appearing in §§ 32–36 of Nörlund's Vorlesungen über Differenzenrechnung [Berlin, 1924]. *D. Moskovitz.*

Stone, William Matthewson. The generalized Laplace transformation with applications to problems involving finite differences. Iowa State Coll. J. Sci. 22, 81–83 (1947).

The transformation $\sum_{n=0}^{\infty}s^{-n-1}F(n)=f(s)$ of functions $F(t)$ is considered. The transformations of the operations of differencing $F(t)$ and of multiplying $F(t)$ by $t(t-1)\cdots(t-m)$ are pointed out. The inverse of transforms and of products of transforms are noted briefly. Those are the operational properties which are needed to solve certain types of linear difference equations by means of the transformation. This paper is an abstract of a thesis; details are not given.

R. V. Churchill (Ann Arbor, Mich.).

Functional Analysis

Vulih, B. Z. Concrete representations of linear partially ordered spaces. Doklady Akad. Nauk SSSR (N.S.) **58**, 733–736 (1947). (Russian)

Let X be a partially ordered linear space of type S_δ (i.e., one satisfying the axioms (I)–(V) of Kantorovich [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–165 (1937)]). The author indicates a proof of the fact that if X has a unit (i.e., an element 1 such that inf $(x, 1)>0$ for all $x>0$) then it is isomorphic to a "normal" subspace of the space $C_\infty(Q)$ of all "generalized" continuous real functions on a bicompact Hausdorff space Q. Here a normal subspace is one which contains x with y whenever $|x|<|y|$ and a generalized continuous function is continuous except for being allowed to take on the values $+\infty$ and $-\infty$ in a nowhere dense subset of Q. The proof indicated leans heavily on the related representation theorem of Krein and Kakutani. The author asserts that in general the space $C(Q)$ of all continuous real-valued functions on a bicompact Hausdorff space Q is of type S_δ if and only if Q has the property that the closure of every open subset is open and that if this is the case $C_\infty(Q)$ is of type S_δ. He also states a theorem to the effect that every space of type S_δ is imbeddable in a certain way in a space with a unit.

In the second paragraph the author states theorems relating upper bounds and (o)-convergence in the space $C_\infty(Q)$ to the pointwise behavior of the functions concerned. In the third paragraph he indicates how his representation theorem may be used to define multiplication and division in spaces of type S_δ. In a final paragraph he states a condition that $C(Q)$ be of type S_δ when Q is the bicompact Hausdorff space associated with a Boolean algebra by Stone's theorem.

G. W. Mackey (Cambridge, Mass.).

Sobolev, V. I. On elements with inverses in partially ordered rings. Doklady Akad. Nauk SSSR (N.S.) **56**, 237–239 (1947). (Russian)

The author considers a commutative ring with identity element which is also a partially ordered Kantorovitch space K_δ [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–165 (1937)]. Also (1) $x^2>0$ for $x\neq 0$ and $x>0$, $y>0$ imply $xy>0$; (2) inf $(x, e)>0$ for all x; (3) if $\{x_n\}$ is a bounded sequence with $x_n\leqq x_{n+1}$, then sup $(x_n y)=(\sup x_n)y$, for every y. An element y is said to be bounded provided $|y|\leqq Me$, for some real $M>0$. It is well known that there exists for each x a resolution of the identity e_λ, $-\infty<\lambda<\infty$, such that x can be represented in the form $x=\int_{-\infty}^{\infty}\lambda de_\lambda$ [H. Freudenthal, Nederl. Akad. Wetensch., Proc. **39**, 645–651 (1936)]. A necessary and sufficient condition for a sequence $\{x_n\}$ to be convergent is that it is bounded and, for every idempotent e' with $0<e'\leqq e$ and every $\epsilon>0$, there exists an idempotent e'' with $0<e''\leqq e'$ such that $-\epsilon e''\leqq(x_m-x_n)e''\leqq\epsilon e''$ for $m, n\geqq n(\epsilon, e')$. That the algebraic operations are continuous follows easily. It is proved that a necessary and sufficient condition for the element $x-\alpha e$ to possess a bounded inverse is that α is a point of constancy for e_λ. It follows that x will possess a bounded inverse if, and only if, $|x|\geqq\alpha e$ for some $\alpha>0$. If $x-\alpha e$ has an inverse (not necessarily bounded) then α is a point of continuity for e_λ.

C. E. Rickart (New Haven, Conn.).

v. Sz. Nagy, Béla. Störungen im Hilbertschen Raume. I. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 755–775 (1942). (Hungarian. German summary)

v. Sz. Nagy, Béla. Störungen im Hilbertschen Raume. II. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 63–79 (1943). (Hungarian. German summary)

Cf. Comment. Math. Helv. **19**, 347–366 (1947); these Rev. **8**, 589.

Fage, M. K. The spectral manifolds of a bounded linear operator in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **58**, 1609–1612 (1947). (Russian)

If the directed boundary of an open half plane in the complex plane makes an angle of $\omega+\pi/2$ with the positively directed real axis, and if the distance from the origin to the boundary, directed along the outward normal, is α, the half plane is denoted by $\Delta(\omega, \alpha)$. If A is a bounded linear operator on a (not necessarily separable) Hilbert space $\mathfrak{E}$, a spectral manifold of A is the set $\mathfrak{E}_A(\omega, \alpha)$ of all those vectors x for which $[\exp \rho(e^{i\omega A-\alpha I})]x\rightarrow 0$ (weakly) as $\rho\rightarrow\infty$. The author describes the elementary properties of spectral manifolds, and establishes the connections between them and the customary spectral theory. The simplest one of these connections is the following assertion: the operator $\mu A-I$ has a bounded inverse R_μ in a subspace $\mathfrak{H}$, regular in the half plane $\Delta(\omega+\pi, -\alpha)$, if and only if $\mathfrak{H}$ is contained in $\mathfrak{E}_A(\omega, \alpha+\epsilon)$ for every $\epsilon>0$.

If, for $\sigma_1\leqq\sigma\leqq\sigma_2$ and $\tau_1\leqq\tau\leqq\tau_2$, $J_1(\sigma)$ and $J_2(\tau)$ are two families of uniformly bounded and mutually commutative idempotent operators which are nondecreasing functions of the parameters σ and τ and such that $(J_1(\sigma)x, x)$ and $(J_2(\tau)x, x)$ are always measurable functions of σ and τ, respectively, the operator N defined by

$$(Nx, y)= (\sigma_2+i\tau_2)(x, y)-\int_{\sigma_2}^{\sigma_1}(J_1(\sigma)x, y)d\sigma-i\int_{\tau_2}^{\tau_1}(J_2(\tau)x, y)d\tau$$

is a generalized normal operator. An operator E is generalized nilpotent if $\mathfrak{E}_E(\omega, \alpha)=\mathfrak{E}$ for every ω and for every $\alpha>0$; this definition is asserted to be equivalent to Gelfand's. If J is a bounded idempotent and if $\mathfrak{H}=\{x: Jx=x\}$ and $\mathfrak{G}=\{x: J^*x=x\}$, the connection among J, $\mathfrak{H}$, and $\mathfrak{G}$ is symbolized by $J=\mathfrak{H}\times\mathfrak{G}$.

If A is a bounded linear operator, write $\mathfrak{H}_1(\alpha)$, $\mathfrak{H}_2(\alpha)$, $\mathfrak{G}_1(\alpha)$, and $\mathfrak{G}_2(\alpha)$ for $\mathfrak{E}_A(0, \alpha)$, $\mathfrak{E}_A(\pi/2, \alpha)$, $\mathfrak{E}_{A^*}(0, \alpha)$, and $\mathfrak{E}_{A^*}(-\pi/2, \alpha)$, respectively. The author's main theorem may be formulated as follows. If $\overline{\mathfrak{H}_i(\alpha)}\subset\mathfrak{H}_i(\alpha+\epsilon)$ and $\mathfrak{E}\ominus\overline{\mathfrak{G}_i(\alpha)}\subset\mathfrak{G}_i(-\alpha+\epsilon)$, for $i=1, 2$, $\epsilon>0$, and $|\alpha|\leqq|A|$; if the idempotents $J_i(\alpha)=\overline{\mathfrak{H}_i(\alpha)}\times\overline{\mathfrak{G}_i(\alpha)}$ are uniformly bounded; and if the quadratic forms $(J_i(\alpha)x, x)$ are measurable functions of α, then $A=N+E$, where N is the generalized normal operator determined by $J_1(\alpha)$ and $J_2(\alpha)$, and E is a generalized nilpotent. The paper contains no proofs.

P. R. Halmos (Princeton, N. J.).

Esser, Martinus. Analyticity in Hilbert space and self-adjoint transformations. Amer. J. Math. **69**, 825–835 (1947).

A detailed proof of the spectral resolution of self-adjoint transformations in Hilbert space, starting from the theorem of Doob and Koopman [Bull. Amer. Math. Soc. **40**, 601–605 (1934)] on the representation of a class of complex analytic functions in the form

$$\int_{-\infty}^{\infty}(\lambda-z)^{-1}d\rho(\lambda)$$

with a real nondecreasing $\rho(\lambda)$. [Another proof on the same lines: B. Lengyel, Acta Litt. Sci. Szeged 9, 174–186 (1939); these Rev. 1, 146.] *B. de Sz. Nagy* (Szeged).

Wei, Telson. A note on linear operations in the metrical space *E*. Acad. Sinica Science Record 2, 36–38 (1947).

The author gives three elementary theorems on continuous linear operators over the Fréchet space of real sequences. The principal theorem states that any such operator is representable as a sequence of continuous linear functionals and the other two theorems apply this result to determining the matrix form of the operator. *R. E. Fullerton.*

Arens, R. F., and Kelley, J. L. Characterizations of the space of continuous functions over a compact Hausdorff space. Trans. Amer. Math. Soc. 62, 499–508 (1947).

Les auteurs donnent deux caractérisations différentes d'un espace de fonctions réelles continues dans un espace compact. De façon précise il s'agit de savoir à quelles conditions un espace de Banach B est isométrique à l'espace de Banach C_X des fonctions réelles continues dans un espace compact X. La première caractérisation fait intervenir le dual B^* de B: il faut et il suffit que pour la sphère unité Σ de B^*: (1) il existe deux hyperplans d'appui de Σ, qui ensemble contiennent tous les points extrémaux de Σ; (2) toute partie de Σ formée de points extrémaux et dont l'adhérence faible ne contient pas de couple de points opposés, est tout entière contenue dans un hyperplan d'appui de Σ. La seconde caractérisation ne fait intervenir que B: il faut et il suffit que: (1) la sphère unité E de B possède un point extrémal; (2) si C est une partie convexe maximale de E, C et $-C$ engendrent E; (3) si une famille de parties convexes maximales de E est telle qu'il n'y ait aucun point de E commun à toutes ces parties, alors pour tout $f\epsilon E$, il existe deux parties C', C'' appartenant à la famille considérée et telles que la somme des distances de f à C' et à C'' soit arbitrairement voisine de 2. L'idée générale des démonstrations est de "réaliser" l'espace X comme un sous-espace de la sphère unité de B^*, muni de la topologie faible. *J. Dieudonné.*

Myers, S. B. Banach spaces of continuous functions. Ann. of Math. (2) 49, 132–140 (1948).

L'auteur considère l'espace de Banach $B(X)$ des fonctions réelles continues et bornées dans un espace topologique; il dit qu'un sous-espace D de $B(X)$ est complètement régulier si pour tout ensemble fermé $K\subset X$ et tout point x_0 du complémentaire de K, il existe $b\epsilon D$ telle que $b(x_0)=\|b\|$, $\sup_{x\epsilon K}|b(x)|<\|b\|$; il cherche des conditions assurant qu'un espace de Banach B soit isométrique à un sous-espace complètement régulier d'un espace $B(X)$. Ses méthodes sont apparentées à la seconde méthode de R. Arens et J. Kelley [voir le précédent compte-rendu]; il considère les cônes de sommet l'origine, engendrés par les parties convexes maximales de la sphère unité de B; pour un tel cône T, il pose, dans B, $F_T(b)=\inf_{t\epsilon T}(\|b+t\|-\|t\|)$; c'est en général une fonction convexe positivement homogène, continue et linéaire dans le sous-espace vectoriel engendré par T. Si chaque T engendre un sous-espace dense dans B, les F_T forment un sous-espace S de la sphère unité de B^*, munie de la topologie faible; si en outre S est réunion de deux ensembles fermés Ω, $-\Omega$ sans point commun, alors B est isométrique à un sous-espace complètement régulier de $B(\Omega)$. L'auteur donne une condition nécessaire très voisine de cette condition suffisante, lorsque X est compact. Il donne enfin la condition nécessaire et suffisante suivante pour que B soit isométrique à un $B(X)$, où X est compact: (1) les F_T doivent être linéaires; (2) B doit posséder un élément e tel que $\|b+e\|=\|b\|+1$ ou $\|b-e\|=\|b\|+1$ pour tout $b\epsilon B$; (3) si $b\epsilon B$, il existe $b'\epsilon B$ tel que $F_T(b')=|F_T(b)|$ pour tout F_T tel que $F_T(e)=1$. *J. Dieudonné* (Nancy).

Theory of Probability

Münzner, Hans. Eine wahrscheinlichkeitstheoretische Behandlung der Jokereigenschaft. Z. Angew. Math. Mech. 25/27, 119–122 (1947). (German. Russian summary)

An urn contains rs numbered balls so that each integer $k\leqq r$ is represented by s balls. In addition, there are $a\geqq 0$ unnumbered balls ("jokers") which may take the place of any number. Balls are drawn without replacement until all r numbers are drawn or sufficiently many jokers are obtained to fill the gaps. The mean and variance of the number of drawings are computed. *W. Feller* (Ithaca, N. Y.).

Moran, P. A. P. The random division of an interval. Suppl. J. Roy. Statist. Soc. 9, 92–98 (1947).

Let $n-1$ points be distributed at random in (0, 1) and denote by $L_1, \cdots, L_n$ the lengths of the n subintervals. The author computes the first four moments of the random variable $S=\sum L_k^2$. He shows that the distribution approaches normality by proving the following theorem. Let $X_1, \cdots, X_n$ be mutually independent with a common distribution and suppose that $E(X_k)=a>0$, $E(X_k^2)=b$, and $\mathrm{Var}\,(X_k^2)=c^2$. Put $U=\sum X_k$ and $V=\sum X_k^2$. Then the variable $c^{-1}n^{\frac{1}{2}}\{-b+na^2V/U^2\}$ is asymptotically normally distributed with mean 0 and variance 1. *W. Feller.*

Finney, D. J. The significance of associations in a square point lattice. Suppl. J. Roy. Statist. Soc. 9, 99–103 (1947).

In a square lattice with a^2 points n points are chosen at random. If two or more of the points are adjacent (counting also diagonals) we speak of *a* doublets, triplets, etc. The present paper reports the means and variances for the numbers of doublets, triplets, and quadruplets observed for $n=2, \cdots, 20$ in an experiment with $a=10$; tables of random numbers were used. It appears that the binomial approximation suggested by H. Todd [same Suppl. 7, 78–82 (1940)] grossly underestimates the variances. *W. Feller.*

Burnens, Ed. Die Erfahrungsnachwirkung bei Wahrscheinlichkeiten. Mitt. Verein. Schweiz. Versich.-Math. 47, 329–352 (1947).

The "after effect" is introduced purely formally by postulating that at time t the probability $p(t)$ of a favorable event is of the form $p(t)=A(t)[A(t)+B(t)]^{-1}$, where

$$A(t)=A(0)+a\int_0^t[1-p(x)]dx,$$

$$B(t)=B(0)-b\int_0^t[1-p(x)]dx$$

with constant $a\geqq 0$, $b\geqq 0$. This leads to a differential equation for $p(t)$ for which the author finds an approximate solution. He then turns to the problem of an urn with w white and r red balls, where after drawings the white balls, but not the red ones, are replaced. He finds formally plausible approximations to the probability q_n of extracting

a white ball at the nth trial. Certain analogies between the two problems are pointed out. *W. Feller* (Ithaca, N. Y.).

Maret, Alfred. De la fonction d'événement d'un ensemble ouvert variable. Mitt. Verein. Schweiz. Versich.-Math. **47**, 321–327 (1947).

Interpretation of the relation between population size, renewal-rate and mortality which leads to the familiar integral equation of renewal theory. *W. Feller.*

***de Toledo Piza, Affonso P. The notion of density of distinct values per interval unit.** São Paulo, 1947. 24 pp.

Let $u_1, \cdots, u_p$ be "p distinct values of a variable u defined within the interval $0 \leqq a < b$." [Presumably the u_k are random variables.] The author begins by proving that $\sum u_k$ cannot converge if $a > 0$. Assuming convergence he introduces a new notion, the "density of distinct values u per interval unit" and claims that $\lim \sum_1^p u_k$ can be expressed as the first moment of this density. Linguistic difficulties prevent a clear understanding of the ideas. *W. Feller.*

Borel, Émile. Sur les développements unitaires normaux. C. R. Acad. Sci. Paris **225**, ~~91~~ 773 (1947).

It is stated without proof that every number x with $0 < x < 1$ admits of a unique representation of the form

$$x = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots,$$

where the a_k are integers with $a_{n+1} \geqq a_n \geqq 2$. If x is chosen at random, a_n is said to be of the magnitude e^n. *W. Feller.*

Lévy, Paul. Remarques sur un théorème de M. Émile Borel. C. R. Acad. Sci. Paris **225**, 918–919 (1947).

The last statement of the preceding review is made more precise by proving that $a_n = \exp[n + X_n n^{1/2}]$, where the random variable X_n is asymptotically normally distributed with mean 0 and variance 1. A similar statement is true for $Q_n = a_1 \cdots a_n$ if the n in the exponent is replaced by $n(n+1)/2$. This asymptotic relation permits also the application of strong laws like the law of the iterated logarithm. *W. Feller* (Ithaca, N. Y.).

Steinhaus, H. Sur les fonctions indépendantes. VII. (Un essaim de points à l'intérieur d'un cube). Studia Math. **10**, 1–20 (1948).

[Paper VI of this series appeared in Studia Math. **9**, 121–132 (1940); these Rev. **3**, 2.] Suppose that $x_k(t)$, $y_k(t)$, $z_k(t)$, $k = 1, \cdots, n$ are the coordinates of n points in a cube. It is supposed that these points move in straight lines at constant velocities and are reflected when they hit the walls. Probability is defined as relative time (over the interval $0 \leqq t < \infty$). Explicit initial points and initial velocities ($t = 0$) are assigned in such a way that the x_k's, y_k's and z_k's are mutually independent, with a common constant probability density. The central limit theorem is then applied to show that the coordinates of the centroid of the system, assigning mass 1 to each point, have, asymptotically as $n \to \infty$, Gaussian distributions. The author stresses that the probability concept is based not on stochastic initial conditions but on the (completely deterministic) evolution of the system with time. *J. L. Doob* (Urbana, Ill.).

Chung, Kai-Lai, and Erdös, Paul. On the lower limit of sums of independent random variables. Ann. of Math. (2) **48**, 1003–1013 (1947).

Les auteurs étudient l'ordre de grandeur de la limite inférieure de sommes $S_n = X_1 + \cdots + X_n$ de variables aléatoires indépendantes et équiprobables. Le résultat fondamental est le suivant. Si la fonction de distribution commune possède (1) une partie absolument continue non identiquement nulle, (2) le premier moment nul, (3) le cinquième moment fini, alors suivant que la série de terme général $1/n\psi(n)$, où $\psi(n) \uparrow \infty$, converge ou diverge, $P\{\liminf_{n\to\infty} n^{1/2}\psi(n)|S_n| = 0\}$ est zéro ou un. Les lemmes utilisés présentent un intérêt propre. *M. Loève.*

Erdös, P., and Kac, M. On the number of positive sums of independent random variables. Bull. Amer. Math. Soc. **53**, 1011–1020 (1947).

Les auteurs utilisent une méthode d'invariance, qui leur est due, pour démontrer le théorème suivant, prouvé par P. Lévy dans le cas binomial. Si le théorème limite central s'applique à la suite des sommes $S_n = X_1 + \cdots + X_n$ de variables aléatoires indépendantes à moyenne 0 et variance 1, alors la distribution limite de la fréquence des S_n positives est donnée par $2\pi^{-1} \sin^{-1} \alpha^{1/2}$ ($0 \leqq \alpha \leqq 1$). La méthode utilisée présente un intérêt par elle-même. *M. Loève.*

Karhunen, Kari. Zur Spektraltheorie stochastischer Prozesse. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **34**, 7 pp. (1946).

Karhunen, Kari. Über lineare Methoden in der Wahrscheinlichkeitsrechnung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **37**, 79 pp. (1947).

The author discusses families of random variables $\{x(t)\}$ with $E\{x(t)\} = 0$ and $E\{|x(t)|^2\} < \infty$. In the following t will be supposed a real parameter, to simplify the review, but many of the author's results are obtained for more general cases. The author defines a measurable process as one for which, if $r(s, t) = E\{x(s)\overline{x(t)}\}$, $r(s, t)$ is for fixed s measurable in t. All the following statements apply to measurable processes. The integral $X = \int_S x(t)dt$ is defined as the uniquely determined random variable satisfying $E\{X\bar{z}\} = \int_S E\{\bar{z}x(t)\}dt$ for every z in the closed (root mean square distance) linear manifold generated by the given variables $\{x(t)\}$. This indirect definition avoids the necessity of discussing the measurability of the individual sample functions which is necessary if X is to be defined as the Lebesgue integral of the individual sample function. A process is said to be of null type if $r(s, t) = 0$ for almost all t, for each fixed s, or equivalently if $\int_S x(t)dt = 0$ with probability 1 for each measurable S. In the stationary case ($r(s, t) = r(t-s)$) it is shown that $x(t)$ can be written in a unique way in the form $x(t) = x_1(t) + x_2(t)$, where the x_1 and x_2 processes are stationary and mutually orthogonal, the first is continuous (that is, its $r(t)$ is continuous) and the second is of null type. The weak law of large numbers involves integral averages of the sample functions and the x_2 component makes no contribution to these averages. If $x(t)$ can be written in the form $x(t) = \int f(t, \lambda)dz(\lambda)$, where the $z(\lambda)$ process has uncorrelated increments ($E\{dz(\lambda_1)\overline{dz(\lambda_2)}\} = 0$, $\lambda_1 \neq \lambda_2$; $= dF(\lambda)$ if $\lambda_1 = \lambda_2 = \lambda$) the integral is said to be a spectral representation. In this case $r(s, t) = \int \overline{f(s, \lambda)} f(t, \lambda) dF(\lambda)$ and conversely if $r(s, t)$ can be written in this form $x(t)$ has a spectral representation; conditions are found under which this representation is uniquely determined. The author uses this theorem to find the spectral representation of a stationary process [Cramér, Ark. Mat. Ast. Fys. **28B**, no. 12 (1942); these Rev. **4**, 13], that of a process of moving averages, and several others. One application is to write $x(t)$ (for t

in a finite interval) in the form $x(t)=\sum z_\nu f_\nu(t)+\eta(t)$, where the η process is of null type, absent if $r(s,t)$ is continuous, the mutually orthogonal f_j's are the characteristic functions of the symmetric kernel $r(s,t)$, and the z_ν's are mutually orthogonal [cf. Kac and Siegert, Ann. Math. Statistics 18, 438–442 (1947); these Rev. 9, 97, who use a special case of this representation]. *J. L. Doob* (Urbana, Ill.).

Gihman, I. I. On a scheme of formation of random processes. Doklady Akad. Nauk SSSR (N.S.) 58, 961–964 (1947). (Russian)

The author discusses random processes whose variables have values in Hilbert spaces. Details of proofs are omitted. In the following Φ and Γ are Hilbert spaces; t is a nonnegative real variable; $x\epsilon\Phi$, $\alpha\epsilon\Gamma$, Γ^e is a set of functions $\alpha(t,x)$. The combination of Γ^e together with a probability measure defined on Γ^e is called a random process. If $f=f[\alpha(t,x)]$, Ef is the expectation (integral) of f, and $E\{\beta(s,y)|_{t'}^{t''};f\}$ is the conditional expectation of f under the hypothesis that $\alpha=\beta(s,y)$ on the interval $[t',t'']$. Finally, if $f\epsilon\Phi$, denote by H the Hilbert space of functions $f=f[\alpha(t,x)]$ with $\|f\|_H=E\|f\|^2<\infty$. The author is particularly interested in solving a generalized differential equation; he wishes to find $x(t)=S[\tau,\xi|\alpha(s,y),t]$ $(x\epsilon H)$ satisfying the initial condition $x(\tau)=\xi$ and the equations

$$\lim_{\Delta\to 0}\Delta^{-1}E\|x(t+\Delta)-x(t)-[\alpha(t+\Delta,x(t))-\alpha(t,x(t)]\|^2=0,$$

$$\lim_{\Delta\to 0}\Delta^{-2}E\|E\{\beta(s,y)|_\tau^t;\ x(t+\Delta)-x(t)$$
$$-[\alpha(t+\Delta,x(t))-\alpha(t,x(t))]\}\|^2=0.$$

He finds conditions under which there is a unique solution. Then $x(t)$ for each t is a point of Φ and P-measure induces a measure in the space of functions $x(t)$, which depends on the initial values ξ and τ. The author sets $\tau=0$ and assumes a probability measure in ξ space Φ thus getting a measure in the space of functions $x(t)$ which defines a new random process, induced by the original one. If the original process satisfies the conditions that $P(A_1A_2)=P(A_1)P(A_2)$ for every pair of Γ^e sets A_1, A_2 defined by restrictions on t in nonoverlapping intervals, the induced process is a Markov process. In this case, and if Φ and Γ are finite-dimensional, with the same number of dimensions, conditions are found that, if f is a numerically valued function, $Ef[x(t)]$ satisfies the Kolmogoroff differential equations in τ and ξ for the transition probabilities of a Markov process.

J. L. Doob (Urbana, Ill.).

Sapogov, N. A. On singular Markov chains. Doklady Akad. Nauk SSSR (N.S.) 58, 193–196 (1947). (Russian)

Let $x_1,\cdots,x_n$ be the random variables of a Markov chain. It is supposed that the x_j's only take on the values ±1 and that p_k' (p_k'') is the probability that $x_{k+1}=1$ if $x_k=1$ (-1). Then if $s_n=\sum_1^n x_k$, Bernstein [Doklady Akad. Nauk SSSR (N.S.) 1928, 55–60] proved that s_n is subject to the central limit theorem if $n^{-\alpha}<p_k'$, $p_k''<1-n^{-\alpha}$, $k=1,\cdots,n$, where $\alpha<\frac{1}{3}$ is a constant. The author proves, using Bernstein's method, that the result is still true if $n^{-\alpha}$ is replaced in the above condition by $\varphi(n)/n^{\frac{1}{3}}$, where $\varphi(n)$ is monotone increasing and becomes infinite with n.

J. L. Doob (Urbana, Ill.).

Blanc-Lapierre, André. Remarques sur les propriétés énergétiques des fonctions aléatoires. C. R. Acad. Sci. Paris 225, 982–984 (1947).

Blanc-Lapierre, André. Sur quelques problèmes posés par la détermination des spectres de puissance ou d'énergie des grandeurs aléatoires. C. R. Acad. Sci. Paris 225, 1264–1266 (1947).

Let $\{X(t)\}$ be the random variables of a stochastic process. It is supposed that $E\{X(t)\}=0$ and that $X(t)$ can be put in the form $X(t)=\int_{-\infty}^{\infty}e^{2\pi i\nu t}dx(\nu)$. Let $d^2\gamma(\nu,\nu')=E\{dx(\nu)\overline{dx(\nu')}\}$ and $dF(\nu)=d^2\gamma(\nu,\nu)$. Let $Y_j(t)=\int_{-\infty}^{\infty}H_j(\nu)e^{2\pi i\nu t}dx(\nu)$ be the result of a linear operation (filter) performed on the $X(t)$'s, and let G_j be the Fourier transform (gain) of H_j. Then for fixed t, τ,

$$(1)\quad \underset{T\to\infty}{\text{l.i.m.}}\ T^{-1}\int_{t-T}^{t}Y_1(\theta)\overline{Y_2(\theta-\tau)}d\theta$$
$$=\int_{-\infty}^{\infty}G_1(\nu)\overline{G_2(\nu)}e^{2\pi i\nu\tau}|dx(\nu)|^2.$$

If one can replace $dx(\nu)$ by $x'(\nu)d\nu$ in the above, then

$$(2)\quad \underset{T\to\infty}{\text{l.i.m.}}\int_{-T}^{T}Y_1(t)\overline{Y_2(t-\tau)}dt=\int_{-\infty}^{\infty}G_1(\nu)\overline{G_2(\nu)}e^{2\pi i\nu\tau}|x'(\nu)|^2d\nu$$

and the latter limit also holds with probability one. Conditions are stated for the limits on the right to be (almost certainly) constant and for the limit in (1) to be a probability-one limit. The averaged versions of these limit equations are also stated. In these the integrands on the left are replaced by their expectations, $T^{-1}\int_{t-T}^{t}$ by $(2T)^{-1}\int_{-T}^{T}$, $|dx(\nu)|^2$ by $dF(\nu)$ and $|x'(\nu)|^2d\nu$ by $\sigma(\nu,\nu)d\nu$ (for the averaged version of (2) the γ distribution is supposed given by a density $\sigma(\nu,\nu')$). In particular, if $Y_1=Y_2=X$ one obtains the instantaneous or mean energy, and if one wishes the energy in a band the filters are chosen to cut off the frequencies outside the band. *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André, et Fortet, Robert. Analyse harmonique des fonctions aléatoires et caractère stationnaire. C. R. Acad. Sci. Paris 225, 1119–1120 (1947).

Let $\{X(t)\}$ be the variables of a stochastic process. Condition (A): $E\{X(t+h_1)\cdots X(t+h_L)\}$ is independent of t for every integer L, and the same is true if one or more of the factors is replaced by its conjugate. Condition (B): $E\{dX(\nu_1)\cdots dX(\nu_L)\}=0$ for every integer L if (*) $\sum_1^L\nu_j\neq0$, and the same is true if one or more of the factors is replaced by its conjugate, where for each such factor the ν_j in (*) is replaced by $-\nu_j$. Then the authors outline a proof that, if $X(t)$ can be put in the form $X(t)=\int_{-\infty}^{\infty}e^{2\pi i\nu t}dx(\nu)$, $X(t)$ satisfies (A) if and only if $X(t)$ satisfies (B). *J. L. Doob.*

Blanc-Lapierre, et Brard. Les fonctions aléatoires stationnaires et la loi des grands nombres. Bull. Soc. Math. France 74, 102–115 (1946).

Let $\{x(t)\}$ be the random variables of a continuous parameter stochastic process, with $-\infty<t<\infty$, $E\{x(t)\}=m(t)$, $R(t_1,t_2)=E\{[x(t_2)-m(t_2)][x(t_1)-m(t_1)]\}$. If $m(t)\equiv0$ and if $R(t_1,t_2)=R(t_2-t_1)$ (stationary case) the authors prove that the strong law of large numbers is applicable whenever, for some $\alpha<1$, $T^{-\alpha}\int_0^T(1-s/T)R(s)ds$ is bounded when $T\to\infty$. This weakens their previous condition that $R(s)$ be Cesàro integrable over $(0,\infty)$ [C. R. Acad. Sci. Paris 220, 134–136 (1945); these Rev. 7, 129]. The present theorem is implied by one of Loève [Revue Sci. 83, 297–303 (1945); these Rev. 8, 38]. In the nonstationary case it is shown that, if $R(t_1,t_2)$ is uniformly continuous, if

$\lim_{t\to\infty} m(t)$ exists and if there is a function $\rho(t)$ such that $\lim_{t_1, t_2\to\infty} [R(t_1, t_2)-\rho(t_2-t_1)]=0$ then $\rho(t)$ is positive definite and many of the theorems applicable to stationary processes (for example, the mean square law of large numbers) are still applicable. *J. L. Doob* (Urbana, Ill.).

✶Feller, W. On probability problems in the theory of counters. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 105–115. Interscience Publishers, Inc., New York, 1948. $5.50.

Soit des événements se produisant à des instants dont la répartition dans le temps obeit au schéma poissonnien classique, de densité donnée a; soit un compteur C ou appareil qui enregistre la production de chacun de ces événements lorsqu'il se produit, si toute fois il n'est pas bloqué à cet instant-là, l'état de blocage étant conditionné par les dâtes des enregistrements précedents selon une loi plus ou moins simple; en désignant par $N(t)$ le nombre aléatoire des enregistrements effectués par C durant un intervalle de temps t, l'auteur donne une méthode pour calculer $E[N(t)]$ et $\sigma[N(t)]$, et en particulier pour obtenir des expressions asymptotiques de ces quantités pour $t\to+\infty$; sa méthode s'applique en particulier au cas où: C est bloqué, après chaque enregistrement qu'il effectue, pendant un temps fixe donné τ; et au cas où: C est bloqué à tout instant t tel que dans l'intervalle $(t-\tau, t)$ il s'est produit au moins un événement; il retrouve ainsi et améliore des résultats obtenus par d'autres auteurs à propos des compteurs Geiger-Müller; à noter que le problème étudié peut être considéré comme un cas particulier, relativement simple, du problème des "appels perdus" en téléphonie automatique.

R. Fortet (Caen).

Hole, N. Note on the statistical analysis of counter data. Ark. Mat. Astr. Fys. **34B**, no. 12, 8 pp. (1947).

The author treats the problem of the best estimate of the decay constants of radioactive sources under various experimental conditions. The finite resolving power of counters is not taken into account. Also the case where Poissonian random events are superimposed on the observations is treated. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

Wisseroth, K. Die günstigste Verteilungsbreite, ein neues Streuungsmass. Z. Angew. Math. Mech. **25/27**, 126–127 (1947).

The shortest interval containing one-half of the observed frequency is proposed as a new measure of dispersion. A graphical solution is outlined and the use of moving averages proposed. No comparison is made with other measures of dispersion. *J. W. Tukey* (Princeton, N. J.).

Baticle, Edgar. Sur une loi de probabilité a priori des paramètres d'une loi laplacienne. C. R. Acad. Sci. Paris **226**, 55–57 (1948).

In statistical estimation of the parameters h and a of the normal law $h\pi^{-\frac{1}{2}}\exp(-h^2(x-a)^2)$ it is, according to the author, usual to assume that h and a are uniformly distributed. From certain assumptions the author concludes that the a priori law $h^{-1}dadh$ is more plausible. It was suggested by M. Dumas [Mémorial Artillerie Française **16**, 599–701 (1937)]. *W. Feller* (Ithaca, N. Y.).

Craig, Allen T. Bilinear forms in normally correlated variables. Ann. Math. Statistics **18**, 565–573 (1947).

The author derives a necessary and sufficient condition for the independence (in the probability sense) of two real symmetric bilinear forms of normally correlated variables. In particular, he shows that a necessary and sufficient condition for independence is that the product of the matrices of the two bilinear forms be zero. He obtains a similar necessary and sufficient condition for the independence of a real symmetric bilinear form and a quadratic form.

S. S. Wilks (Princeton, N. J.).

Olmstead, Paul S., and Tukey, John W. A corner test for association. Ann. Math. Statistics **18**, 495–513 (1947).

The following nonparametric test for the association of two continuous variables is proposed. In the scatter diagram, draw the sample median lines $x=x_m$, $y=y_m$ and label the four quadrants $+, -, +, -$ according to the sign of $(x-x_m)(y-y_m)$. Count, in order of decreasing x, along the observations until forced to cross the horizontal median; let k_1 be the number of observations met before this crossing, provided with the sign of the quadrant in which they lie. Repeat this process moving up from below, to the right from the left, and down from above. The "quadrant sum" $S=k_1+k_2+k_3+k_4$ of the numbers thus obtained yields the test; it evidently gives special weight to the extreme values of the variables. The distribution of S in the case of independent (x, y) is tabulated for the sample sizes 2, 4, 6, 8, 10, 14 and ∞. Extensions to higher dimensions and to serial correlation are discussed and illustrated. *G. Elfving.*

Daniels, H. E. Grouping corrections for high autocorrelations. Suppl. J. Roy. Statist. Soc. **9**, 245–249 (1947).

The author gives the relation between the moment μ_{ij} of a bivariate frequency function $f=f(x_1, x_2)$ and the moment m_{ij} $(i, j=0, 1, 2)$ of the discrete distribution G obtained from f by grouping. The characteristic function of G is expressed as a double Fourier series whose terms involve the characteristic function of f. An approximation, $1-(\omega^2/\sigma^2)c$ (valid when Sheppard's corrections are valid), is given for the correlation ρ between X_1 and X_2, where X_1-X_2 is normal, X_1 and X_2 have the same distribution, all grouping intervals have the same length ω and the values of the centers of the initial intervals are the same; c depends on $C=(m_2-m_{11})/\omega^2$ $(m_2=m_{20}=m_{02};\ \sigma^2=\mu_{20}=\mu_{02})$. By means of a fourth-order Gram-Charlier series for the distribution of X_1-X_2, the effect, on the approximation, of a moderate amount γ_2 of kurtosis is studied. A table of c for $C=.01(.01).30$ (for $\gamma_2=0, 1$) is given, showing that c is in error by at most 8.9 per cent when normality is assumed but in fact $\gamma_2=1$. *D. F. Votaw, Jr.* (New Haven, Conn.).

Vaswani, S. P. A pitfall in correlation theory. Nature **160**, 405–406 (1947).

An example of x and y distributed on a two-branched curve with (a) the marginal distributions normal, (b) the correlation coefficient positive (greater than 0.3), (c) y decreasing as x increases. *J. W. Tukey* (Princeton, N. J.).

Maurin, Jacques. Un mode de calcul général de la fonction de probabilité de moyennes. C. R. Acad. Sci. Paris **225**, 1268–1269 (1947).

By quantizing a continuous variable to enable the use of combinatorial arguments, it is shown that the frequency function of the mean of p random variables has a formula

which leads, by passage from the discrete to the continuous, to the usual expression for the density function of the mean as an infinite convolution integral. A "generalization" to the mean of means, which differs in small details from the form of the above, for reasons not clear to the reviewer, is also given. *J. Riordan* (New York, N. Y.).

Maurin, Jacques. Extension analytique d'un calcul de la fonction de probabilité de moyennes correspondant à une probabilité négative. C. R. Acad. Sci. Paris 226, 51–53 (1948).

It is pointed out that the formula for the density function of the mean of p random variables, given in the article reviewed above, may be restated for non-normalized variables [$\varphi(x)$ replaces $f(x)$, where $f(x)$ is a density function and $f(x)=\varphi(x)/\int_{-\infty}^{\infty}\varphi(x)dx$] and used for evaluation of certain multiple infinite integrals. *J. Riordan.*

Egudin, G. I. Certain relations between the moments of the distribution of extreme values in random samples. Doklady Akad. Nauk SSSR (N.S.) 58, 1581–1584 (1947). (Russian)

Let $X_1, \cdots, X_n$ be identically and independently distributed real chance variables. Let $Y_1, \cdots, Y_n$ be the chance variables obtained from the X's by rearranging them in ascending order. The author proves that $nE(X_1^r)=\sum_{i=1}^n E(Y_i^r)$. *J. Wolfowitz* (New York, N. Y.).

Allard, Georges. Détermination de la valeur la plus probable des grandeurs statistiques. I. Généralités. J. Phys. Radium (8) 8, 212–214 (1947).

The probability distribution of the random variable X depends on several parameters α_j. Out of n observations it is known that n_k were in the interval (x_k, x_{k+1}) $(k=1, \cdots, r, \sum m_k=1)$. The author discusses formally the problem of estimating the α_j. The point of departure is Bayes's rule. The author seems unaware of the modern literature on statistical estimation. *W. Feller.*

Geppert, M. P. Mutungsgrenzen und Mutungswahrscheinlichkeit. Z. Angew. Math. Mech. 25/27, 253–263 (1947). (German. Russian summary)

A discussion of the foundations of the theory of statistical estimation. "In the beginning, Gini's objections against the statistical test criteria are refuted, and the logical contents and limits of the tests of significance are analyzed. By transferring by steps the reasoning to the problem of estimation of a parameter in a population from a sample, the author tries to bring the concepts 'confidence limits' and 'fiducial distribution' into harmony, and to explain their logical significance and coherence." *W. Feller.*

Tukey, John W. Non-parametric estimation. II. Statistically equivalent blocks and tolerance regions—the continuous case. Ann. Math. Statistics 18, 529–539 (1947).

[For part I, by Scheffé and Tukey, cf. the same Ann. 16, 187–192 (1945); these Rev. 7, 21.] This paper extends exact tolerance distributions to the case of a continuous cumulative distribution function. The method makes use of Wald's device for successive elimination in the construction of tolerance regions in multidimensional cases, using for the partition process an ordered set of arbitrary real-valued functions of the essentially arbitrary underlying chance quantity, subject only to the requirement that these functions shall each have a continuous cumulative distribution. *G. W. Brown* (Ames, Iowa).

Wishart, John. Proof of the distributions of χ^2, of the estimate of variance, and of the variance ratio. J. Inst. Actuaries Students' Soc. 7, 98–103 (1947).

Hoel, Paul G. Discriminating between binomial distributions. Ann. Math. Statistics 18, 556–564 (1947).

Let $x_1, \cdots, x_k$ be k independent random variables, each being the number of successes in n trials from a binomial distribution with probability p of success in a single trial. Let s denote the statistic $n_0\sum_{i=1}^k(x_i-\bar{x})^2/\{\bar{x}(n_0-\bar{x})\}$. Let χ_α^2 be the $100(1-\alpha)$ percent point of the distribution of χ^2 with $k-1$ degrees of freedom. If $\bar{x}$ and $n_0-\bar{x}$ are large, an approximate best critical region of size α, independent of p, for testing the hypothesis $H_0: n=n_0$, against the alternative $n>n_0$, is obtained by rejecting H_0 if and only if $s>\chi_\alpha^2$. The test also has approximate optimum properties for deciding between a binomial distribution with $n=n_0$ and a Poisson distribution. Approximate confidence limits for n are obtained. [The misprint e exp [ ] for exp [ ] occurs several times.] *T. E. Harris* (Santa Monica, Calif.).

Walsh, John E. On the power efficiency of a t-test formed by pairing sample values. Ann. Math. Statistics 18, 601–604 (1947).

Let x_i and y_i, $i=1, \cdots, n$, independent variates, be distributed normally with means μ and ν and equal variances σ^2. To test the difference in means, statistic t_1 or statistic t_2 may be used, $\bar{x}$ and $\bar{y}$ being the sample means,

$$t_1=\frac{\{\bar{x}-\bar{y}-(\mu-\nu)\}\{n(n-1)\}^{\frac{1}{2}}}{\{\sum_{i=1}^n[x_i-y_i-(\bar{x}-\bar{y})]^2\}^{\frac{1}{2}}},$$

$$t_2=\frac{\{\bar{x}-\bar{y}-(\mu-\nu)\}\{n(n-1)\}^{\frac{1}{2}}}{\{\sum_{i=1}^n(x_i-\bar{x})^2+\sum_{i=1}^n(y_i-\bar{y})^2\}^{\frac{1}{2}}},$$

where t_1 is distributed as Student's t with $n-1$ degrees of freedom and t_2 with $2n-2$ degrees of freedom. Since the most powerful test for one-sided or symmetrical two-sided tests is based on t_2, the author computes the power function and power efficiency of t_1 for one-sided alternatives at the .05, .025 and .01 levels of significance, $n\geqq 4$. *L. A. Aroian* (New York, N. Y.).

Kimball, Bradford F. Some basic theorems for developing tests of fit for the case of the non-parametric probability distribution function. I. Ann. Math. Statistics 18, 540–548 (1947).

This paper contains results on the distributions of various functions of successive differences of $F(X_i)$ $(i=1, \cdots, n)$, where $X_1, \cdots, X_n$ (which are in nondecreasing order) are drawn independently at random from a continuous cumulative distribution function $F(x)$. The suitability of some of these functions for testing goodness of fit is discussed. Let $u_r=F(X_r)-F(X_{r-1})$ $(r=2, \cdots, n)$, $u_1=F(X_1)$ and $u_{n+1}=1-F(X_n)$. The main result is that

$$(1)\quad E\left\{\prod_{j=1}^h u_{r_j}^{p_j}\right\}=\Gamma(n+1)\prod_{j=1}^h\Gamma(p_j+1)/\Gamma(n+1+p_1+\cdots+p_h),\quad h\leqq n+1,$$

where $(r_1, \cdots, r_h)$ is any set of distinct integers from the set of integers $1, \cdots, n+1$, and $p_j>-1$ $(j=1, \cdots, h)$. By means of (1) there are obtained: (i) an upper bound on the quantity in (1); (ii) the cumulative distribution function of $F(X_{k+q})-F(X_k)$ $(k=0, 1, \cdots, n;\ q\leqq n+1-k)$, where $F(X_0)=0$ and $F(X_{n+1})=1$; (iii) low moments and the moment generating function of $y_m=\sum_s u_s^2$, where the range of s is a subset, consisting of m distinct integers, of the set

of integers $1, 2, \cdots, n+1$. The cumulative distribution function in (ii) is the same as that of $F(X_q)$: this result is similar to a result of Wilks [same Ann. 12, 91–96 (1941); these Rev. 3, 9]. The use of y_m is tentatively proposed for testing goodness of fit; arguments are advanced for choosing $m=n+1$. It is stated that the asymptotic distribution of y_m "appears to be normal" and that the distribution of $\{\sum_1^{n+1}[u_i-(n+1)^{-1}]^2\}^{\frac{1}{2}}$ "takes on the normal character rapidly with increasing n." *D. F. Votaw, Jr.*

Armitage, P. Some sequential tests of Student's hypothesis. Suppl. J. Roy. Statist. Soc. 9, 250–263 (1947).

Let μ and σ^2 be, respectively, the unknown mean and variance of a normal population on which independent observations are made. The discussion separates into two parts. (a) One-sided tests. It is desired to test H_0, that $\mu=0$, against H_1, that $\mu/\sigma=D>0$. For samples of fixed size the best test is of course based on Student's t. This is compared with a binominal Wald sequential test based on the chance variable Y, defined as follows. Let X be normally distributed, with parameters μ and σ^2. Then $Y=0$ or 1 according as $X\leqq 0$ or $X>0$. A comparison of expected sample sizes is made. It turns out that, in spite of the obvious loss of information, the sequential test requires on the average fewer observations when H_0 is true, and roughly about the same number when H_1 is true.

(b) Two-sided tests. Here one wishes to decide among H_0, H_1, and H_2, that $\mu/\sigma=-D$. The author proposes to use the binomial sequential test described above simultaneously to decide between H_0 and H_1, and H_0 and H_2. Then H_0 is adopted if both tests decide in its favor. Otherwise either H_1 or H_2 is adopted, according as H_2 or H_1 is the first hypothesis rejected. No inconsistent results can occur. The average sample number for this test is unknown, as is also the average sample number of Wald's test [Ann. Math. Statistics 16, 117–186 (1945), pp. 183–186; these Rev. 7, 131]. The author tries to compare the two by a series of experiments. The reviewer is not inclined to believe that these experiments shed much light on the problem.

J. Wolfowitz (New York, N. Y.).

Albert, G. E. A note on the fundamental identity of sequential analysis. Ann. Math. Statistics 18, 593–596 (1947).

The condition $|\varphi(t)|\geqq 1$ given by Wald [same Ann. 17, 493–497 (1946); these Rev. 8, 284] for validity of his identity $E\{e^{Z_n t}[\varphi(t)]^{-n}\}=1$ and for the validity of its differentiation with respect to t under the expectation sign is removed. *A. M. Mood* (Santa Monica, Calif.).

***Chrząszcz, Roman. Ein Problem der Bestimmung und Eliminierung von systematischen Beobachtungsfehlern.** Sammlung wissenschaftlicher Arbeiten der in der Schweiz internierten Polen, Band 1, Heft 3, pp. 23–26. Eidg. Kommissariat für Internierung und Hospitalisierung, 1943.

Suppose that k groups of observations l_{ij} are made of n quantities w_i ($i=1, \cdots, n$; $j=1, \cdots, k$), each observation being subject to a systematic error s_j independent of i but different in different groups of observations, and also subject to random errors z_{ij}. Then $l_{ij}=w_i+s_j+z_{ij}$. The author shows how the systematic errors s_j may be determined from the observations. He further shows how the systematic errors s_j may be eliminated and the mean random error determined. *W. E. Milne* (Corvallis, Ore.).

Sales, Francisco. Some considerations on the foundations of the theory of errors. Revista Mat. Hisp.-Amer. (4) 7, 165–172 (1947). (Spanish)

Mathematical Biology

***Luneburg, Rudolf. Metric methods in binocular visual perception.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 215–240. Interscience Publishers, Inc., New York, 1948. $5.50.

In this paper the author's approach differs somewhat from that given in his book [Mathematical Analysis of Binocular Vision, Princeton University Press, 1947; these Rev. 9, 49]. It is assumed that there exists a psychometric distance function $D(P, Q)$ for any two objects P, Q, and certain axioms are imposed on D: these include the triangle inequality and axioms that the visual space is finitely compact and convex. An assumption of free movability leads to the conclusion that the visual space is a Riemannian space of constant curvature K, so that a polar coordinate system ρ, ϕ', θ' exists such that the psychometric line element is given by $ds^2=(d\rho^2+\rho^2 d\phi'^2+\rho^2\cos^2\phi' d\theta'^2)(1+\frac{1}{4}K\rho^2)^{-2}$. On the basis of experimental results, the author sets up relations $\rho=f(\gamma)$, $\phi'=\phi$, $\theta'=\theta$ between the above coordinates and the three FH coordinates γ, ϕ, θ described in the cited review. [Correction: for $RLP+LRP$ read $RLP-LRP$.] The function $f(\gamma)$ is for the moment arbitrary; it is determined by further appeal to observation. It is asserted that iseikonic transformations (γ increased by a constant, with ϕ, θ unchanged) are conformal transformations. This leads to the line element in visual space:

$$ds^2=4(e^{\sigma\gamma}+Ke^{-\sigma\gamma})^{-2}(\sigma^2 d\gamma^2+d\phi^2+\cos^2\phi d\theta^2),$$

where σ, K are personal constants of the observer. Some specific applications are considered. [If the sensation of distance is based solely on binocular vision, as the author supposes throughout, then the sensation of distance, and all optical illusions based on that sensation, should be lacking in a monocular observer. This is not the case, and for that reason the reviewer regards the author's theory as incomplete.] *J. L. Synge* (Pittsburgh, Pa.).

***de Toledo Piza, Affonso P. On an integral equation of interest in the theory of the movement of a population.** São Paulo, 1947. 4 pp. (Portuguese)

Consider a variable population and let $H(b)$ be its size, $\phi(t)\cdot H(b)$ the rate of increase, and assume that $H(t)p(z, t)$ individuals living at time t will still be living at time $z>t$. The author points out that

$$H(z)=H(t)p(z, t)+\int_t^z H(x)\phi(x)p(z, x)dx.$$

W. Feller (Ithaca, N. Y.).

***de Toledo Piza, Affonso P. On the factor of elimination $p(t, \lambda)$ and its applications to the study of the movement of a population.** São Paulo, 1947. 9 pp. (Portuguese)

Under certain assumptions the function $p(z, b)$ of the preceding review will satisfy the functional equation $p(z, t)=p(x, t)p(z, x)$, whence one finds by logarithmic differentiation that $p(z, t)=f(z)g(t)$. *W. Feller.*

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 502–506 (1947).

The author sets $\epsilon\equiv y_0-y$, where y_0 defines an extremal in the calculus of variations and y defines a neighboring curve, invokes the normal law $\varphi(\epsilon)$, and notes that φ is maximized at $\epsilon=0$. The point of these manipulations is not clear, at least to the reviewer. *A. S. Householder.*

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 718–724 (1947).

Continuation of the paper reviewed above. The calculus of variations enters through a least-square minimization; the extremals represent possible "modes of life" for the species. Tensorial relations are noted and extension in phase defined. No attempt is made to illustrate the usefulness of the concepts. *A. S. Householder* (Oak Ridge, Tenn.).

TOPOLOGY

Zarankiewicz, K. Sur les relations symétriques dans l'ensemble fini. Colloquium Math. **1**, 10–14 (1947).

The author proves the following theorem. Let G be a graph of n vertices and assume that the order of each vertex is at least K. [The order of a vertex is the number of edges incident to it.] Then if $K>n(p-2)/(p-1)$, G contains a complete subgraph of p vertices, i.e., G contains p vertices any two of which are connected by an edge. The result is false for $K\leqq n(p-2)/(p-1)$. The author expresses the theorem in a different language: he speaks of symmetric relations instead of graphs. [Reviewer's remark. The author's result can be deduced from Turán's, Mat. Fiz. Lapok **48**, 436–452 (1941); these Rev. **8**, 284.]

P. Erdös (Syracuse, N. Y.).

Tutte, W. T. The factorization of linear graphs. J. London Math. Soc. **22**, 107–111 (1947).

A graph N of even order n is said to be prime if it contains no set of $\frac{1}{2}n$ branches which together use up all the nodes $a_1, a_2, \cdots, a_n$. Let S denote a subset consisting of f of these n nodes. If we suppress the nodes S and all branches belonging to them, what is left of N will, in general, consist of several disconnected pieces. Let h_u denote the number of these pieces that are of odd order. The author proves that a graph N is prime if and only if it contains an S such that $h_u>f$.

The principal tool is the Pfaffian $P=\sum \epsilon c_{ij}c_{kl}\cdots c_{rs}$, summed over all partitions of the integers from 1 to n into pairs $ij, kl, \cdots, rs$. The symbols c_{ij} $(i<j)$ are independent indeterminates corresponding to the branches a_ia_j of N, but we write $c_{ij}=0$ whenever nodes a_i and a_j are not directly joined, and $c_{ji}=-c_{ij}$. The coefficient ϵ is ±1 according as $ijkl\cdots rs$ is an even or odd permutation of $12\cdots n$. Thus N is prime if and only if P vanishes identically.

H. S. M. Coxeter (Toronto, Ont.).

Wuytack, F. Les transformations bicontinues d'un espace topologique. Simon Stevin **25**, 199–200 (1947).

A correction to the author's earlier note [same vol., 142–145 (1947); these Rev. **9**, 98]. *R. Arens.*

Hu, Sze-tzen. Archimedean uniform spaces and their natural boundedness. Portugaliae Math. **6**, 49–56 (1947).

In a uniform space X with a uniform structure U, a set B is called bounded if for any R in U there is an integer n such that $B\times B\subset R^n$. It is proved that, when X satisfies the obvious generalization of 0- (or Cantor-) connectedness, then (1) if R belongs to U and the sets A, B are bounded, then $A\cup B$, B^- and $R(B)$ are bounded; (2) total boundedness implies boundedness. For Banach spaces, using the metric structure, the present boundedness coincides with the usual notion. The connection between the present boundedness and that defined in convex topological linear spaces is not considered. It is pointed out that this boundedness is not the notion to which one is led by the "generalized-metric" approach to uniform spaces. *R. Arens.*

Hu, Sze-tsen. A group multiplication for relative homotopy groups. J. London Math. Soc. **22**, 61–67 (1947).

Für die relativen Homotopiegruppen $\pi_{k+1}(Y, Y_0)$, $k>0$, eines Raumes Y modulo einer Teilmenge Y_0 von Y wird ein Produkt definiert, welches jedem $\alpha\epsilon\pi_{n+1}(Y, Y_0)$ und jedem $\beta\epsilon\pi_{m+1}(Y, Y_0)$ ein Element $\alpha\circ\beta\epsilon\pi_{m+n}(Y, Y_0)$ zuordnet; beim Randhomomorphismus der Gruppen $\pi_{k+1}(Y, Y_0)$ in die absoluten Homotopiegruppen $\pi_k(Y_0)$ geht es in das von J. H. C. Whitehead [Ann. of Math. (2) **42**, 409–428 (1941); diese Rev. **2**, 323] eingeführte Produkt über. *B. Eckmann.*

Hu, Sze-tsen. An exposition of the relative homotopy theory. Duke Math. J. **14**, 991–1033 (1947).

Although a large amount of knowledge has accumulated about the homotopy groups of Hurewicz, this is the first organized account of the topic. Both the absolute and relative homotopy groups are defined and their basic group properties established. The "homotopy sequence" of a pair (Y, Y_0) is proved to be exact, and is shown to be a covariant functor under mappings. The operations of $\pi^1(Y_0)$ on $\pi^n(Y, Y_0)$ are defined and the question of simplicity is studied. The Hurewicz theorem is proved in the relative case: $\pi^n(Y, Y_0)\approx H^n(Y, Y_0)$ if (Y, Y_0) is r-aspherical for $r<n$, and is n-simple. The Hurewicz isomorphisms relating the groups of Y to those of certain function spaces over Y are extended to the relative case. Indeed the isomorphisms provide an isomorphism of the homotopy sequences. The paper does not consider the products defined by J. H. C. Whitehead [Ann. of Math. (2) **42**, 409–428 (1941); these Rev. **2**, 323] or the torus homotopy groups of R. H. Fox [Proc. Nat. Acad. Sci. U. S. A. **31**, 71–74 (1945); these Rev. **6**, 279]. *N. E. Steenrod* (Princeton, N. J.).

Chern, Shiing-shen. On the characteristic ring of a differentiable manifold. Acad. Sinica Science Record **2**, 1–5 (1947).

Let $H=H(n, N)$ be the Grassmann manifold of n-planes through O in Euclidean space E^{n+N}. An imbedding of a differentiable manifold M in E^{n+N} defines a mapping f of M into H. This induces a homomorphism f^* of the cohomology ring $R(H)$ of H into that of M. The ring $f^*(R(H))$ is called the characteristic ring of M; its elements, characteristic classes. The latter include the Stiefel-Whitney classes W^k. The ring $R(H)$ is studied with the help of Schubert chains. Theorem: the classes W^k generate $f^*(R(H))$. The formula for ring products leads to nonimbedding theorems. For instance, if n is even and is not of the form $2(2^k-1)$, $k\geqq1$, then projective n-space cannot be imbedded in E^{n+2}. Proofs are only sketched. *H. Whitney.*

★Hopf, H. Zur Topologie der komplexen Mannigfaltigkeiten. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 167–185. Interscience Publishers, Inc., New York, 1948. $5.50.

Necessary conditions of a topological nature are given which an even dimensional orientable manifold M must satisfy in order that M admit the structure of a complex analytic manifold. In the tangent bundle of M there must exist a transformation which to each tangent vector L at $m\epsilon M$ assigns another tangent vector at m which is not parallel to L. A tangent vector field in M is then considered with a single singularity, the index of the singularity being the Euler characteristic C of M. Applying the transformation, a 2-field is obtained also with a single singularity. The index of this singularity is an element of a suitable homotopy group and is proved to be divisible by C. Following a detailed analysis of this singularity it is proved that the spheres S^4 and S^8 are not complex manifolds. The complex analytic structure of a manifold determines an orientation. It is shown that in the complex projective plane this orientation is always the usual one regardless of the particular complex analytic structure.

S. Eilenberg (New York, N. Y.).

Kirchhoff, Adrian. Sur l'existence de certains champs tensoriels sur les sphères à n dimensions. C. R. Acad. Sci. Paris **225**, 1258–1260 (1947).

A manifold V^n is called of type J if for each p in V there is a nonsingular linear transformation T_p of the tangent space at p with no (real) proper vector, T_p being continuous in p. This is certainly true if V has a complex structure. It is shown that if the sphere S^n is of type J, then the group L_{n+2} of nonsingular linear transformations of $(n+2)$-space R^{n+2} has a simply transitive family: with fixed e, there is a nonsingular linear transformation L_x for each x with $L_xe=x$, L_x being continuous in x. It follows easily that S^{n+1} has a parallelism. Because of results of B. Eckmann [Comment. Math. Helv. **15**, 1–26 (1943); these Rev. **4**, 173] and G. W. Whitehead [Ann. of Math. (2) **43**, 132–146 (1942); these Rev. **3**, 197] we must have $n=2$ or $8k+6$.

H. Whitney (Cambridge, Mass.).

Nordon, Jean. Les éléments d'homologie des quadriques et des hyperquadriques. Bull. Soc. Math. France **74**, 116–129 (1946).

The author determines the homology groups and fundamental groups of the nondegenerate quadrics in real projective space, of the nondegenerate hyperquadrics (given by a Hermitian form) in complex projective space, and of the corresponding objects in quaternion projective space. [Partial results in this direction have been announced by Steenrod and Tucker, Bull. Amer. Math. Soc. **47**, 399–400 (1941).] The groups depend of course on the signature of the defining quadratic, or Hermitian, form. Subdivisions into cells are employed. In the real case two subdivisions are set up; one is based on the fact that every quadric (except the sphere) has a product of two spheres as double covering; the other consists of an increasing sequence of rectilinear generators and subquadrics such that the difference of two consecutive terms is homeomorphic with an open element. One of these subdivisions permits the derivation of Betti numbers and torsion coefficients, but does not give the cycles; the other gives the cycles, but does not give the Betti numbers and torsion coefficients, because the incidence numbers cannot be determined in advance. Complex and quaternion quadrics are also treated by subdivision into cells, corresponding to the second type of subdivision for the real case. All the cells are actual cycles and, in contrast to the real case, there is no need for another type of subdivision.

H. Samelson (Ann Arbor, Mich.).

Eckmann, Beno. On infinite complexes with automorphisms. Proc. Nat. Acad. Sci. U. S. A. **33**, 372–376 (1947).

The methods of the author's earlier note [same Proc. **33**, 275–281 (1947); these Rev. **9**, 244] are applied to yield the following theorem. Let K be a locally finite complex, acyclic (using finite chains) in all dimensions less than N. Let G be a group of automorphisms of K without fixed cells and with a finite fundamental domain. Then the cohomology groups $\mathcal{H}^n(K, J)$ obtained using finite cochains over a group J are, for $n<N$, isomorphic to groups $\Pi^n(G, J)$ determined by G and J. The group $\Pi^N(G, J)$ is isomorphic to the subgroup $\mathcal{H}_0^N$ of $\mathcal{H}^N$ determined by the finite cocycles which are coboundaries of infinite cochains. This leads to the following theorem for a closed orientable manifold M of dimension m: if the homotopy groups $\pi_n(M)$ vanish for all $1<n<N$, where $m/2<N\leqq m$, then $\pi_N(M)\cong\Pi^{m-N}(G, J)$, where $G=\pi_1(M)$ and $J=$integers.

S. Eilenberg (New York, N. Y.).

Whitehead, J. H. C. On the groups $\pi_r(V_{n,m})$ and sphere-bundles. Corrigendum. Proc. London Math. Soc. (2) **49**, 479–481 (1947).

In the original paper [same Proc. (2) **48**, 243–291 (1944); these Rev. **6**, 279] the groups $\pi_r(V_{n,m})$ were calculated for $r=n-m+1$ and $r=n-m+2$. However, some of these results were found to be in disagreement with previous results of Eckmann [Comment. Math. Helv. **14**, 141–192, 234–256 (1942); **15**, 318–339 (1943); these Rev. **3**, 317, 318; **5**, 104] and G. W. Whitehead [Ann. of Math. (2) **43**, 132–146 (1942); these Rev. **3**, 197]. In this note the author finds his mistake [in the proof of theorem 5, p. 480] and recalculates the groups. The revised table of groups is in agreement with the results of Eckmann and G. W. Whitehead.

R. H. Fox (Princeton, N. J.).

Eilenberg, Samuel. On a linkage theorem by L. Cesari. Bull. Amer. Math. Soc. **53**, 1192–1195 (1947).

In Euclidean three-space E consider the set M consisting of the three axes X, Y, Z. Let δ be a positive number and let N be the set consisting of the four lines X': $(y=0, z=-\delta)$, Y': $(x=0, z=\delta)$, Z': $(x=\delta, y=\delta)$, Z'': $(x=-\delta, y=-\delta)$ [X' and Y' intersect the plane strip bounded by the lines Z' and Z'']. L. Cesari [Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6**(**75**), 267–291 (1942); these Rev. **8**, 258], in connection with his work on Lebesgue area of surfaces, has proved the following linkage theorem, which reveals a property of Euclidean 3-space. Any continuous and closed curve C of $E-M$, of which the points have a distance greater than 2δ from M, is contractible in $E-M$, if and only if C is contractible in $E-N$. In the present paper the author, observing that the theorem can be reformulated as the problem of isomorphism of the fundamental groups of $E-M$ and $E-N$, gives a new and concise proof of the theorem.

L. Cesari (Bologna).

Keldyš, L. V. Continuous images of compacta. Doklady Akad. Nauk SSSR (N.S.) **58**, 181–184 (1947). (Russian)

A decomposition of a metric set F into a finite number of closed summands is an ϵ-decomposition if no summand has

a diameter exceeding ϵ. If F is of dimension n, then the decomposition is called regular if the dimension of the intersection of every k of these summands, for whatever k, is at most $n-k+1$. Let X and Y be closed subsets of some Euclidean space and Y a continuous image of X. The author proves the lemma: if there exists an ϵ-decomposition of Y, of order m, then there exists a regular decomposition of X which induces in Y an ϵ-decomposition of order m. The proof is inductive, beginning with $m=0$. From this, it follows that, in order that the image Y of a compact X be of dimension m, it is necessary and sufficient that m be the smallest integer such that for every ϵ there is a regular decomposition of X which induces in Y an ϵ-decomposition of order m. *L. Zippin* (Flushing, N. Y.).

Young, Gail S., Jr. A characterization of 2-manifolds. Duke Math. J. **14**, 979–990 (1947).

This characterization of the 2-dimensional manifolds is novel in that it includes manifolds with a boundary and at the same time makes no explicit mention of an exceptional point set ultimately to be recognized as constituting this boundary. Let M be a nondegenerate, locally compact, locally connected, connected metric space with no local cut points. The author shows that, if for each point x of M there is a neighborhood U of x such that every simple closed curve in U separates M, then M is a 2-manifold. Actually a slightly stronger form of the theorem is proved. Turning next to considerations in the plane the author gives an enlightening critique on the interrelations between the axioms 0–8 of R. L. Moore [Foundations of Point Set Theory, Amer. Math. Soc. Colloquium Publ., v. 13, New York, 1932]. By way of application a new proof is given of the theorem of G. T. Whyburn [Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942, p. 197; these Rev. **4**, 86] that the image of a 2-manifold under a light interior transformation is a 2-manifold. *W. W. S. Claytor* (Washington, D. C.).

GEOMETRY

***Levi, Beppo. Leyendo a Euclides. [Reading Euclid].** Editorial Rosario, Rosario, 1947. 225 pp.

This pleasant little book records in informal fashion some thoughts of a mathematician occasioned by a reading of Euclid's Elements. Though the author disclaims any intention of writing a serious historical study or a modern critique of Euclid, there is much of both in the book. While not every reader will be convinced by the author's attempts at interpreting away some of Euclid's deficiencies, the arguments presented are always interesting. After a discussion [68 pages] of pre-Euclidean philosophy (geometry and Socratic thought) on which the author bases many of his later conclusions concerning interpretations of passages in the Elements, the book commences the reading of the Elements, dividing the commentary into the following chapters: (I) Definitions, postulates, common notions; the theory of equality; (II) The angle-sum of a triangle and the fifth postulate; (III) Geometric algebra and the theory of proportions; (IV) The method of exhaustion; (V) The arithmetic books. A short bibliographical note ends the book. *L. M. Blumenthal* (Columbia, Mo.).

Kasner, Edward. Neo-Pythagorean triangles. Scripta Math. **13**, 43–47 (1947).

The author considers triangles whose lengths of sides a, b, c are complex numbers and calls the triangles neo-Pythagorean if $a^2+b^2+c^2=0$. If a triangle is neo-Pythagorean, then $\cos A\cdot\cos B\cdot\cos C=-1$, $\cos 2A+\cos 2B+\cos 2C=3$, $\cos A/a+\cos B/b+\cos C/c=0$, $\sin^2 A+\sin^2 B+\sin^2 C=0$, $m_a:m_b:m_c=a:b:c$, $h_a^{-2}+h_b^{-2}+h_c^{-2}=0$, where the letters have the usual meanings in elementary geometry. For most of these results the converses hold with minor restrictions. The author assumes without discussion that all concepts and formulae of Euclidean triangle geometry and trigonometry remain valid and meaningful when applied mechanically to the neo-Pythagorean triangles. Since the term Pythagorean is generally used for right triangles whose sides are integers, and this restriction is not made in the present paper, the name neo-Pythagorean may be misleading. *A. J. Kempner* (Boulder, Colo.).

Gardner, G. H. F. Geometry of the Kasner triangle. Amer. Math. Monthly **54**, 579–583 (1947).

The Kasner plane consists of the totality of points (x, y) for which the distance between any two points is $M_{12}=(x_2-x_1)^2/(y_2-y_1)$. The linear-element is $ds=dx^2/dy$. The motion group is the three-parameter group G_3: $X=mx+h$, $Y=m^2y+k$. The Kasner plane is perhaps the simplest example of a Finsler space which is not merely Riemannian. This plane first arose in the investigation of the conformal geometry of horn angles by Kasner. If x is interpreted as the curvature of a side of a horn angle and y as the rate of change of the curvature per unit length of arc, then M_{12} is the conformal measure of the horn angle. The author studies the geometry of a general triangle in the Kasner plane. Many properties have been established by Kasner, Comenetz, DeCicco and Ladue. In the present article, the parabolic and hyperbolic analogues of the nine-point circle and the analogue of the Simpson line are discussed in detail. *J. DeCicco* (Chicago, Ill.).

Haantjes, J., and Seidel, J. The congruence order of the elliptic plane. Nederl. Akad. Wetensch., Proc. **50**, 892–894 = Indagationes Math. **9**, 403–405 (1947).

It is announced in this note that a congruence order of the elliptic plane $E_{2,r}$ (with respect to the class of metric spaces) is seven; that is, any metric space is congruent with a subset of $E_{2,r}$ whenever each seven of its points has that property. Since the elliptic plane contains an equilateral sextuple [Blumenthal, Trans. Amer. Math. Soc. **59**, 381–400 (1946); these Rev. **8**, 82] this result is the best possible. The paper gives only a brief outline of the methods used to establish this result; detailed proofs are to be published later in the thesis of J. Seidel.

The reviewer notes that the definition given for metric space must be strengthened to read [condition 1] $pq=0$ if and only if $p=q$. The reviewer feels, moreover, that the symmetric property of the metric is not necessarily implied by what is stated, even if condition 1 is so amended. It is observed that the authors define two metric spaces to be congruent if there exists a distance-preserving mapping of each into the other. This is not in agreement with the sense in which "congruence" has become established in the litera-

ture (that is, a distance-preserving mapping of one space onto another) for a distance-preserving mapping of each of two metric spaces into the other is not necessarily even a homeomorphism. It seems desirable to point out that the sketch of the proof of theorem 1, case A, may give the impression that the theorem is established by using only the imbeddability in $E_{2,r}$ of the four septuples containing the orthocentric quadruple, since the argument given makes no mention of other septuples. But this would be counter to an example constructed by L. M. Kelly [who, using similar methods, proved in his Missouri dissertation that a congruence order of $E_{2,r}$ with respect to the class of semimetric spaces is eight], and a communication from one of the authors informs the reviewer that the full proof does indeed make use of the assumed congruent imbedding of each septuple in the plane. Typographical error: the authors refer to the elliptic plane as having total curvature r^2 instead of $1/r^2$. *L. M. Blumenthal* (Columbia, Mo.).

Verriest, G. The bisection of the triangle in non-Euclidean geometry. Simon Stevin **25**, 162–164 (1947). (Dutch)

The author explains how to find, without any appeal to continuity, a line dissecting a given triangle into two parts of equal area. The essential step is to replace Saccheri's isosceles birectangle by a centrally symmetrical quadrangle of the same area. *H. S. M. Coxeter* (Toronto, Ont.).

Lipka, Stephan. Über das Eulersche Dreieck der Bolyaischen Geometrie. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 83–91 (1941). (Hungarian. German summary)

Der Eulersche Satz der Euklidischen Geometrie, nach dem der Höhenschnittpunkt eines Dreiecks, der Mittelpunkt des Umkreises und Schwerpunkt in einer Geraden liegen, gilt in der hyperbolischen Geometrie nicht. Die genannten Punkte bilden das Eulersche Dreieck des gegebenen Dreiecks. Es wird der folgende Satz bewiesen. Es sei ABC ein rechtwinkliges hyperbolisches Dreieck. Es seien $\mathfrak{a}$ und $\mathfrak{b}$ die hyperbolischen Längen der Katheten von ABC. Gilt nun $1>\operatorname{th}\mathfrak{b}/\operatorname{th}\mathfrak{a}\geqq\frac{1}{3}\sqrt{3}$, so liegen die Ecken des Eulerschen Dreiecks von ABC in einer Abstandslinie (Hypercykel). *From the author's summary.*

Mikusiński, Jan G.-. Sur la notion de point remarquable dans la géométrie du triangle. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **1**, 41–44 (1946). (French. Polish summary)

The author proposes to clarify the notion of a "remarkable point" of the geometry of the triangle. Those points are "remarkable" whose barycentric coordinates are given by three homogeneous functions of the sides of the triangle, the functions fulfilling certain conditions, or postulates, set up for the purpose. The coordinates of known remarkable points, like the orthocenter or the circumcenter, satisfy the specified conditions. No attempt is made to use those conditions to obtain new remarkable points exhibiting some geometrical properties of interest. *N. A. Court.*

Mikusiński, Jan G.-. Sur quelques propriétés du triangle. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **1**, 45–50 (1946). (French. Polish summary)

Let A', B', C' be points marked on the sides BC, CA, AB of a given triangle ABC so that $BA':A'C=p$, $CB':B'A=q$, $AC':C'B=r$, where p, q, r are given quantities. The author finds the ratio of the area of the triangle $A'B'C'$ to the area of ABC, in terms of p, q, r. The same is done for the area of the triangle formed by the three lines AA', BB', CC'. [The first of those two ratios is known. See, for instance, Nouvelle Correspondance Math. **6**, 472–474 (1880).] *N. A. Court* (Norman, Okla.).

Sáenz García, Clemente. Interesting aspects of the theory of parallelohedra. Revista Acad. Ci. Madrid **36**, 296–306 (1942). (Spanish)

The author compares the four principal kinds of "parallelohedra," each of which, with an infinity of equal and similarly situated replicas, would fill the whole space without interstices [Fedorov, Verh. Mineralog. Ges. St. Petersburg [Zapiski Imp. S.-Peterburg. Mineralog. Obščestva] (2) **21**, 1–279 (1885), pp. 193–198; Tutton, Crystallography and Practical Crystal Measurement, 2d ed., v. 1, London, 1922, pp. 567, 723]. These are the triparallelohedron or cube (6 faces), the tetraparallelohedron or hexagonal prism (8 faces), the hexaparallelohedron or rhombic dodecahedron (12 faces), and the heptaparallelohedron or truncated octahedron (14 faces). He computes the ratio $A:V^{\frac{2}{3}}$, where A and V are the surface and volume. This ratio progressively decreases in the four cases, though remaining considerably greater than its value for the sphere. [On p. 306, $\sqrt{36\pi}$ is a misprint for $\sqrt[3]{36\pi}$.] He also computes the ratio $E:V$, where E is the volume of the inscribed sphere. This density of the corresponding packing of spheres might be expected to increase with the complexity of the cell. But, as is well known, its value is actually greater for the rhombic dodecahedron than for the truncated octahedron. This is not surprising when we notice that, although the rhombic dodecahedron has only twelve faces, every one of them provides a contact of the insphere with a neighboring sphere, whereas, of the fourteen faces of the truncated octahedron, only eight provide contacts of spheres. The paper is illustrated with excellent drawings. *H. S. M. Coxeter.*

Bottema, O. The prismoid. Simon Stevin **25**, 153–161 (1947). (Dutch)

A prismoid is a polyhedron in which two n-gons, whose sides are respectively parallel, are joined by n trapezoids. When $n=3$ this is a pyramidal frustum, by Desargues' theorem; but when $n>3$ it may be more complicated. Let G and B be the areas of the two basic n-gons, while M is the area of the parallel section midway between them. Then the section dividing the altitude in the ratio $\lambda:\mu$ (where $\lambda+\mu=1$) has area

$$(G\lambda-B\mu)(\lambda-\mu)+4M\lambda\mu=G\lambda^2+(4M-G-B)\lambda\mu+B\mu^2.$$

The author gives a new proof of the statement by Minkowski [Math. Ann. **57**, 447–495 (1903), p. 463 = Gesammelte Abhandlungen, v. 2, pp. 230–276 (p. 245)] that this quadratic form is indefinite. Consequently, for a convex prismoid, $2\sqrt{M}\geqq\sqrt{G}+\sqrt{B}$, with equality holding only when the two basic n-gons are homothetic (so that the prismoid reduces to a pyramidal frustum). *H. S. M. Coxeter.*

Thébault, Victor. Sphères de Tücker d'un polyèdre harmonique. C. R. Acad. Sci. Paris **226**, 305–306 (1948).

Klug, L. Ueber Tetraeder, deren Kanten eine Kugel berühren. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 23–35 (1942). (Hungarian. German summary)

The author considers three concurrent lines a, b, c tangent to a sphere. The sphere intersects the faces of the trihedral angle formed by these lines in three circles. Let $XYZX'Y'Z'$ be any hexagon, with the vertices X and X' on a, Y and Y'

on b, Z and Z' on c, whose opposite sides are tangents to one of the three circles. It is shown that the opposite vertices of the hexagon are elements of three involutions on the edges of the trihedral angle. The double points of these involutions determine two tetrahedra whose edges are tangent to the given sphere. If the edges of a tetrahedron are tangents to a sphere, the points of tangency are located on four circles of the sphere. It is shown that the edges of such a tetrahedron intersect four other circles of the sphere perpendicularly and thereby determine a second tetrahedron. The two tetrahedra and one of the polar tetrahedra of the sphere form a desmic system. *E. Lukacs.*

Rados, Gustav. Über die Gleichung des durch 3 seiner Punkte bestimmten Kreises und der durch 4 ihrer Punkte bestimmter Kugel. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 1–8 (1941). (Hungarian. German summary)

***Yates, Robert C. A Handbook on Curves and Their Properties.** J. W. Edwards, Ann Arbor, Mich., 1947. x+245 pp. $3.25.

This is a convenient source of information about special plane curves. The contents are arranged alphabetically according to some two dozen individual curves or classes of curves and almost as many topics (such as curvature, evolutes, pedal curves, sketching). Under each topic will be found notes on history, definitions, equations, special properties, geometrical and mechanical constructions, etc., as well as carefully drawn figures. *R. P. Boas, Jr.*

Lorent, H. Transformations de courbes planes. I. Anais Fac. Ci. Pôrto **26**, 5–20 (1941).

A tout point P du plan, l'auteur associe le point Q, tel que: (I) la droite OQ fait l'angle constant α avec le diamètre conjugué de OP par rapport à une conique fixe de centre O; et (IIa): la droite PQ est parallèle à une droite donnée, ou (IIb): la droite PQ passe par un point fixe. Cette transformation quadratique, étudiée par la géométrie analytique élémentaire, donne des modes de génération de diverses courbes spéciales usuelles. *P. Belgodère* (Paris).

Lorent, H. Transformations de courbes planes. II. Anais Fac. Ci. Pôrto **26**, 65–83 (1941).

Conservant la condition (I) de l'article précédent, la droite PQ est maintenant astreinte à faire avec OP, ou avec OQ, un angle fixe β. Cette transformation cubique, directe ou inverse, permet d'engendrer de façon simple de nombreuses courbes spéciales. *P. Belgodère* (Paris).

Toscano, Letterio. Una proprietà della concoide de Nicomede. Anais Fac. Ci. Pôrto **26**, 204–205 (1941).

La transformation ponctuelle plane (multivoque), dans laquelle deux points correspondants A, A' sont alignés avec le centre O d'une circonférence fixe C, le milieu de AA' étant sur C, transforme évidemment toute droite D du plan en une conchoïde de Nicomède de pôle O, dont l'asymptote est symétrique de D par rapport à O. *P. Belgodère.*

Bosteels, G. The scyphoid, a rational quartic curve. Nieuw Tijdschr. Wiskunde **35**, 121–125 (1947). (Dutch)

The equation is $x^4-y^4+4rxy^2=0$.

Deaux, R. Sur la focale de van Rees. Nieuw Tijdschr. Wiskunde **35**, 151–155 (1947).

The curve is the locus of the foci of conics inscribed in a quadrilateral.

Evans, H. P. Volume of an n-dimensional sphere. Amer. Math. Monthly **54**, 592–594 (1947).

Baer, Reinhold. Projectivities of finite projective planes. Amer. J. Math. **69**, 653–684 (1947).

Das Studium der projektiven Abbildungen einer endlichen ebenen Geometrie auf sich wird weitläufig, ohne die koordinatenmässige Darstellung über einem Galoisfeld heranzuziehen. Mit rein abzählenden Methoden gelangt Verfasser nach Voranstellung allgemeiner Sätze über projektive Abbildungen einer endlichen Ebene auf sich als Permutation ihrer Punkte und unter Heranziehung von Congruenzen zwischen der Ordnung und dem Charakter einer Permutation zu einer Klassifizierung der Projektivitäten, deren Ordnung die Potenz einer Primzahl p ist. In diesem Falle ergeben sich Zusammenhänge zwischen der Struktur der Fixelemente der Projektivität und einfachen Congruenzen, die von p und der Anzahl n der Punkte auf jeder Geraden abhängen. Besonderes Interesse bieten die Gruppen von Projektivitäten, von denen keine ausser der Identität ein eigentliches Viereck invariant lässt. Die Ordnung der Gruppe ist dann ein Teiler von $(1+n+n^2)(1+n)n^3(1-n)^2$; ist sie ein Teiler eines der vier Faktoren, so haben in jedem Falle alle Projektivitäten der Gruppe eine einheitliche Struktur der Fixelemente, die sich angeben lässt. *R. Moufang* (Frankfurt am Main).

***Baer, Reinhold. The infinity of generalized hyperbolic planes.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 21–27. Interscience Publishers, Inc., New York, 1948. $5.50. have

Eine verallgemeinerte hyperbolische Ebene ist eine Ebene mit projektiver Verknüpfung, in der eine Polarkorrelation existiert, die eine Einteilung der Elemente der Ebene in 3 teilerfremde Klassen gestattet gemäss folgenden Axiomen: ein Element ist dann und nur dann absolut, wenn es mit seinem entsprechenden Element inzident ist; ein Punkt ist dann und nur dann innere Punkt, wenn seine Polare äussere Gerade ist; jede Gerade durch einen inneren Punkt ist innere Gerade; Geraden, die in der Polarkorrelation konjugiert sind, schneiden sich in einem inneren Punkt. Dann folgt durch indirekten Schluss, unter Heranziehung von Sätzen über Polarkorrelationen in endlichen Geometrien, dass die Klasse der inneren Elemente nicht leer ist und die Anzahl der inneren Punkte auf einer inneren Geraden nicht endlich ist. *R. Moufang* (Frankfurt am Main).

Barrau, J. A. A semi-regular configuration in the plane. Nieuw Tijdschr. Wiskunde **35**, 93–96 (1947). (Dutch)

This is a configuration $(6_5+10_3, 15_{2+2})$, consisting of six "Arabic" points, ten "Roman" points, and fifteen lines, in the projective plane. Each Arabic point lies on five of the lines, each Roman point on three; and each line contains two points of each kind. The author shows that these incidences are invariant under a group of order 120, isomorphic with the symmetric group of degree 5, and that the alternating subgroup consists of collineations. But he fails to notice that the fifteen lines may be regarded as a section of the fifteen planes of symmetry of the icosahedron, while the Arabic and Roman points are sections of the pentagonal and trigonal axes of rotation. Taking the plane of section perpendicular to one of the pentagonal axes, he could have obtained a metrical specialization of the configuration in the following symmetrical form. Five of the Roman points are the vertices of a regular pentagon which is stellated to

form a pentagram having five Arabic vertices. These vertices belong also to a larger pentagon which is stellated to form a large pentagram whose vertices are the remaining five Roman points. The sixth Arabic point is the center of the figure. [See Coxeter, Regular Polytopes, London, 1948, p. 91, fig. 6.2c with five extra lines drawn through the center or p. 66, the last part of fig. 4.5A, where the Arabic and Roman points are marked 0 and 2, respectively.] The sixty collineations of the configuration correspond to the rotations of the icosahedral group, while the remaining sixty permutations arise from the isomorphism that relates the icosahedron and great icosahedron. [Coxeter, op. cit., p. 106.] Finally, it should be observed that the fifteen lines and the ten Roman points (without the Arabic points) form a Petersen graph. *H. S. M. Coxeter* (Toronto, Ont.).

v. Szökefalvi Nagy, Gyula. Konjugierte Polygone in der projektiven Ebene. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 61, 441–459 (1942). (Hungarian. German summary)

Let $A_1, \cdots, A_N$ be N points in the projective plane and assume that no two points of this set are collinear. A projective polygon is obtained by connecting the points A_kA_{k+1} $(k=1, 2, \cdots, N; A_{N+1}\equiv A_1)$. The points A_k and A_{k+1} determine two line segments; one is a side of the polygon, the other (complementary) segment is called a complementary side of the polygon. By replacing some of the sides of a polygon P by their complementary sides a new polygon is obtained which is said to be conjugate to P. If all sides of a polygon are replaced by their complements a polygon P', the complement of P, is obtained. The N points determine a family of 2^N conjugate polygons. Consider an N-sided polygon or broken line P. Let n (i) be the greatest (smallest) number of elements which P may have in common with any line. The number n is called the order, the number i the index and $n-i$ the defect of P. The points of intersection of two nonadjacent sides of a polygon are called double points. A point collinear with k sides is considered as a double point of multiplicity $\frac{1}{2}k(k-1)$. The genus of a polygon of order n is given by $p=\frac{1}{2}(n-1)(n-2)-d$, where d is the number of double points.

The author proves a number of theorems on projective polygons. A few examples are as follows. (1) Complementary polygons have the same defect. (2) Among the 2^N polygons conjugate to a convex N-gon there are $2(N+1)$ with defect 2; the rest have the defect 4. (3) An N-gon has at most $\frac{1}{2}N(N-3)$ double points. (4) Consider a convex N-gon P and k $(2\leqq k\leqq N-2)$ sides of P which form q disconnected broken lines. The conjugate polygon, obtained by replacing these k sides by their complements, has $\frac{1}{2}k(k-3)+q$ double points and genus $2k-q$. *E. Lukacs.*

Convex Domains, Extremal Problems

Aleksandrov, A. D. The method of gluing in the theory of surfaces. Doklady Akad. Nauk SSSR (N.S.) 57, 863–865 (1947). (Russian)

In the following let F be an abstractly given convex surface. First let F be homeomorphic to a circular disk and such that its boundary L is concave towards F (a generalization of the requirement that the integral over the geodesic curvature over any subarc of L with F "to the left" is nonnegative). Then F is isometric to a convex surface F' in E^3 with a plane boundary L' and such that L' is the boundary of the projection of F' on the plane of L'. If F is homeomorphic to a disk then it is isometric to a geodesically convex domain on a complete convex surface in E^3, if and only if its boundary is concave towards F. If L is a quasi-geodesic on F (that is, L is concave towards both sides) and p is a point of L, then a convex surface F' in E^3 exists which is isometric to a convex neighborhood U of p on F, such that the image L' of the subarc of L in U is a limit of geodesic segments on regular convex surfaces in E^3 which tend to F'. A geodesic polygon on F with given sides and angles less than π bounds a maximal arc if and only if it is isometric to the lateral surface of a convex pyramid in E^n with the same data (it is assumed that at least one polygon with these data on F exists). *H. Busemann.*

Busemann, Herbert. Note on a theorem on convex sets. Mat. Tidsskr. B. 1947, 32–34 (1947).

If a point set is convex and closed we shall say that it has the property C; if to any point P of the space there is exactly one point of the set nearest to P, we shall say that the set has the property J. B. Jessen proved [Mat. Tidsskr. B. 1940, 66–70; these Rev. 2, 261] that in Euclidean spaces the properties C and J are equivalent. As a generalization of this result the author gives a necessary and sufficient condition for the structure of a straight line space in which C involves J and a second condition for the space in which J involves C. A straight line space is a metric space through any two points of which a "straight line" (i.e., a continuous curve defined by certain natural postulates) can be drawn. *L. Fejes Tóth* (Budapest).

Rédei, L. Über die Stützebenenfunktion konvexer Körper. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 60, 64–69 (1941). (Hungarian. German summary)

Für den bekannten Satz von Minkowski über die karakteristischen Eigenschaften der Minkowskischen Stützebenenfunktionen konvexer Körper [s. Bonnesen und Fenchel, Theorie der konvexen Körper, Ergebnisse der Math., Bd. 3, H. 1, Springer, Berlin, 1934, S. 164] wird ein einfacher Beweis mitgeteilt, in dem die geometrische Bedeutung des Verfahrens klar hervortritt. *Author's summary.*

Fejes, László. Die regulären Polyeder, als Lösungen von Extremalaufgaben. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 61, 471–477 (1942). (Hungarian. German summary)

The author proves the following results. Let $P_1, \cdots, P_n$ be n points on the unit sphere. Let $d(P_1, \cdots, P_n)$ denote the smallest distance between any two of them. For $n=4$, 6 and 12 the maximum of $d(P_1, \cdots, P_n)$ is assumed for the corresponding regular polyhedra, but for $n=8$ and 20 the cube and dodecahedron are not the solutions of the extremal problem. In the introduction some other extremal problems for polyhedra are discussed. *P. Erdös* (Syracuse, N. Y.).

Fejes, László. Über die izoperimetrische bzw. izepiphane Eigenschaft der Ellipse bzw. des Ellipsoids. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 62, 88–94 (1943). (Hungarian. German summary)

Let K be a closed convex curve, T its area, λ its affine arc length. Let T_n be the minimum of the areas of circumscribed n-gons and t_n the maximum of the areas of inscribed n-gons. The author proves that

$$(1)\quad \lim_{n\to\infty} n^2(T_n-t_n)=3\lim_{n\to\infty} n^2(T_n-T)=\tfrac{3}{2}\lim_{n\to\infty} n^2(T-t_n)=\tfrac{1}{4}\lambda^3.$$

The author also proves the 3-dimensional analogue of (1), and some other analogous results. *P. Erdös.*

Bottema, O. Which ellipse through four given points has the smallest area? Nieuw Tijdschr. Wiskunde 35, 126–140 (1947). (Dutch)

To answer his question, the author employs affine coordinates, taking the four points to be $(p_1, 0)$, $(p_2, 0)$, $(0, q_1)$, $(0, q_2)$. The conic

$$\left(\frac{x}{p_1}+\frac{y}{q_1}-1\right)\left(\frac{x}{p_2}+\frac{y}{q_2}-1\right)+\frac{2\lambda xy}{p_1p_2q_1q_2}=0$$

is an ellipse if $\lambda^2<p_1p_2q_1q_2$, and its area is stationary if λ satisfies a certain cubic equation. This equation is found to have three real roots, only one of which satisfies the inequality; so the problem has a unique solution. In particular, if the four points form a parallelogram, the smallest ellipse is the one for which the diagonals are conjugate diameters. [Since this is an affine property of the parallelogram, it can alternatively be established by taking the parallelogram to be a square; but there is no such simple method for the general convex quadrangle.] *H. S. M. Coxeter* (Toronto, Ont.).

***Shiffman, Max. On the isoperimetric inequality for saddle surfaces with singularities.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 383–394. Interscience Publishers, Inc., New York, 1948. $5.50.

It is known that the isoperimetric inequality $4\pi A\leqq L^2$, where A is the area enclosed by a simple closed curve of length L, holds for regions on regular surfaces with curvature $K\leqq 0$. The author shows that it is also valid for certain classes of surfaces with singularities. First saddle polyhedra are considered, i.e., simply connected polyhedra with the intrinsic property that the sum of the face angles emanating from an interior vertex is always at least 2π. By means of an elementary discussion of the geodesics of the polyhedron the above inequality is reduced to the case of the plane. A similar method is applicable to saddle polyhedral surfaces, i.e., surfaces composed of a finite number of curved faces with curvature $K\leqq 0$ such that the sum of the geodesic curvatures of an interior edge with respect to the two adjacent faces is at most 0 and that the sum of the face angles at each interior vertex is at least 2π. Finally the inequality is proved for harmonic surfaces in a Euclidean space R^m by varying the surface in an R^{m+2} containing R^m so that possible singularities disappear. *W. Fenchel.*

Algebraic Geometry

***Weil, André. Foundations of Algebraic Geometry.** American Mathematical Society Colloquium Publications, vol. 29. American Mathematical Society, New York, 1946. xix+289 pp. $5.50.

Advances in the more arithmetic branches of modern algebra and their application to number theory naturally lead, as we may venture to say today, to problems which to the well-informed mathematician either appeared familiar as part of the heritage of classical algebraic geometry or seemed to be intrinsically adapted to a solution by more conceptual geometric methods. Furthermore, since major parts of the theory of algebraic functions of one variable had been fitted into the system of algebra it was sensible that similar interpretations and attempts at solutions were (and had to be) tried for higher dimensional problems. In order to understand and appreciate the ultimate significance of this book the reader may well keep in mind the preceding twofold motivation for the interest in algebraic geometry. Classical algebraic geometry made free use of a type and mode of reasoning with which the modern mathematician often feels uncomfortable, though the experience based on a rich and intricate source of examples made the founders of this discipline avoid serious mistakes in final results which lesser men might have been prone to make. The main purpose of this treatise is to formulate the broad principles of the intersection theory for algebraic varieties. We find those fundamental facts without which, for example, a good treatment of the theory of linear series would be difficult. The doctrine of this book is that an unassailable foundation (and thereby justification) of the basic concepts and results of algebraic geometry can be furnished by certain elementary methods of algebra. Thus, the reader will agree after some time that he is finding a delicate tool which can serve him to remove the traces of insecurity which occasionally accompany geometric reasoning. Incidentally, the term "elementary" used here and by the author is to be understood in a restricted technical sense, in the sense that general ideal theory and the theory of power series rings are not brought into play too often. The proofs require the general plan of using the "principles of specialization," as formulated algebraically by van der Waerden; and they are by no means elementary in the customary connotation. To some readers the adherence to a definite type of approach, where another author may have deemed it more instructive or appropriate to use slightly different methods, may tend to cloud occasionally immediate understanding by the less adept. However, once the reader has grasped the real geometric meaning of a definition or theorem (he then has to forget occasionally the fine points resulting from the facts that the author imposes no restriction on the characteristic of the underlying field of quantities) he will recognize how skilfully the language and methods of algebra are used to overcome certain limitations of spatial intuition.

The author begins his work with judiciously selected results from the theory of algebraic and transcendental extensions of fields [chapter I, Algebraic preliminaries]. Special emphasis has to be placed on inseparable extensions, which incidentally means a more complete account than is found in books on algebra. The further plan of the book is perhaps best appreciated if one starts to ponder over a more or less heuristic definition of "algebraic variety," and then asks one's self informally how one should define "intersections with multiplicities" of "subvarieties." Then, in view of the principle of local linearization in classical analysis, the author's arrangements of topics is more or less dictated by the ultimate subject under discussion, provided one does not place the interpretation of geometrical concepts by ideal theory at the head of the discussion. Therefore the technical definitions of point, variety, generic point and point set attached to a variety [chapter IV, The geometric language] must be preceded by suitable algebraic preparations [essentially in chapter II, Algebraic theory of specializations] and more arithmetic studies [chapter III, Analytic theory of specializations]. Crucial results in this connection, based on arithmetical considerations, are found in proposition 7 on page 60 and theorem 4 on page 62, where the existence of a well-defined multiplicity is proved for specializations.

For further work, the author next introduces the concept of simple point of a variety in affine space by means of the linear variety attached to the point. [See the significant propositions 19 to 21 on pages 97–99.] Next, the intersection theory of varieties in affine space is presented through the following stages of increasing complexity: (i) intersection with a linear subspace of complementary dimension, the 0-dimensional case, with the important criterion for multiplicity 1 in proposition 7 on page 122, and ultimately the criterion for simple points in theorem 6 on page 136; (ii) intersection with a linear subspace of arbitrary dimension, with theorem 4 on page 129 which justifies the invariant meaning of the term "intersection multiplicity of a variety with a linear variety along a variety" [chapter V, Intersection multiplicities, special case]. In chapter VI, entitled General intersection theory, the results for the linear case are extended so as to culminate in the important theorem 2 on page 146 concerning the proper components of the intersection of two subvarieties in a given variety. Furthermore, all important properties of intersection multiplicities are established. Later, in appendix III, it is shown that the properties established for a certain symbol are characteristic for intersection multiplicities and uniquely define that concept. It may be mentioned that the topological definition of the chain intersections on manifolds coincides with the algebraically defined concept of this book. Of course, the underlying coefficient field has to be the field of all complex numbers and further simplifying assumptions on the variety have to be made. However, this comparison cannot be made at the level of chapters V and VI, since there one deals with affine varieties to which the ordinary topological considerations are not directly applicable.

The subsequent chapter VII, Abstract varieties, provides the necessary background for the aforementioned connections and also contains complete proofs of those results which one might have formulated first had one deliberately adopted ideal-theoretic intentions at an early stage. The abstract varieties of this chapter are obtained by piecing together varieties in affine spaces by means of suitably restricted birational transformations. This definition of the author has turned out to be very fruitful for the work on the Riemann hypothesis for function fields and the study of Abelian varieties in general. In the course of the work, the results of the preceding chapters are extended so as to lead up to the important theorem 8 on page 193 related to Hopf's "inverse homomorphism." The chapter ends with a theory of cycles of dimension s, that is, formal integral combinations of simple abstract subvarieties of dimension s. The notion of the intersection product of cycles is also introduced here [page 202], by means of which the investigation of equivalence theories can be initiated. This is done more explicitly in chapter IX, Comments and discussion; apparently the Riemann-Roch theorem for surfaces should now be accessible to a careful re-examination. As a further result, the theory of quasi-divisibility of Artin and van der Waerden is developed in theorems 3 and 4 on pages 224–225 and theorem 6 on page 230. These theorems exhibit the relations between the theory of cycles of highest dimension and the theory of quasi-divisibility, where naturally some of the results in appendix II, Normalization of varieties, are to be added for the necessary integral closure of the required rings of functions. In this appendix the author relates his results on the normalization of algebraic varieties to those of Zariski. At this point the individual reader may well compare the elementary and the ideal-theoretic approach to a group of theorems. In appendix I, Projective spaces, often used properties and facts concerning projective spaces are quickly developed on the basis of the preceding work. This brief discussion not only deals with results which are generally useful in algebraic geometry, but also contains one of the theorems on linear series of divisors which was frequently used in the classical work [see page 266]. Because of the wealth of material and the excellent "advice to the reader" prefacing this rich and important book the reviewer feels that he should mention some of the highlights and not delve into a discussion of technical details. In short, the only way to appreciate this treatise is actually to read it.

O. F. G. Schilling (Chicago, Ill.).

Goddard, L. S. Prime ideals and postulation formulae. Proc. Cambridge Philos. Soc. **44**, 43–49 (1948).

Dans un travail antérieur [Proc. Cambridge Philos. Soc. **39**, 35–48 (1943); ces Rev. **4**, 168], l'auteur a déterminé la base de l'idéal premier associé à une sous-variété irréductible V_{d-1} d'une variété V_d de Segre ou de Veronese, dans le cas où elle est la section complète de V_d par une hypersurface. Dans le présent travail, il donne explicitement la base dans le cas général. Si $\mathfrak{P}$ est l'idéal premier associé à une variété irréductible M de type $(\lambda_1, \cdots, \lambda_r)$ sur une variété de Segre V_d, et si $\mu=\max(\lambda_1, \cdots, \lambda_r)$, alors $\mathfrak{P}=(\mathfrak{p}, \mathfrak{a})$ où $\mathfrak{p}$ est l'idéal de V_d et $\mathfrak{a}$ un idéal dont la base est constituée de formes canoniques toutes de degré μ.

La formule de postulation de la variété de Segre V_d pour les hypersurfaces d'ordre ρ est $\chi(\rho)=C_{n_1+\rho}^{n_1} \cdots C_{n_2+\rho}^{n_2}$, valable pour toutes les valeurs possibles de ρ. Si Y est la veronesienne représentant les hypersurfaces d'ordre h dans S_n, et si sur Y, N représente une hypersurface d'ordre r de S_n, l'idéal premier de N est tel que $\mathfrak{P}=(\mathfrak{p}, F_1 \cdots F_\sigma)$ où $\sigma=C_{n+\rho}^{\rho}$ et les F_i sont des formes canoniques de degré α; ρ et α étant définis par $r=\alpha h-\rho$ $(\rho<h)$. La postulation de Y pour les hypersurfaces d'ordre ρ est $C_{n+\rho h}^{n}$. En appliquant ces résultats concurremment avec les formules de Severi pour le genre arithmétique, l'auteur retrouve et généralise certaines formules de Room [The Geometry of Determinantal Loci, Cambridge University Press, 1938, p. 471].

L. Gauthier (Nancy).

Châtelet, François. Essais de géométrie galoisienne. Bull. Soc. Math. France **74**, 69–86 (1946).

Exposition of the "Galois methods" exploited in the author's thesis [Ann. Sci. École Norm. Sup. (3) **61**, 249–300 (1944); these Rev. **7**, 323] and some later papers.

D. B. Scott (London).

Lesieur, Léonce. Les problèmes d'intersection sur une variété de Grassmann. C. R. Acad. Sci. Paris **225**, 916–917 (1947).

The author enunciates a precise rule for expressing the product of two Schubert symbols, of dimensions ρ and ρ', as the sum of Schubert symbols of dimensions $\rho+\rho'$. Corresponding to the Schubert symbol $(a_0, a_1, \cdots, a_p)$, where $0\leqq a_0<a_1<\cdots<a_p\leqq n$, $\rho=a_0+\cdots+a_p-\frac{1}{2}p(p+1)$, there is a partition $\{\lambda_0, \lambda_1, \cdots, \lambda_p\}$ of ρ into $p+1$ nonnegative parts in nonascending order of magnitude, where $\lambda_k=n-p-a_k+k$, and $\lambda_0=n-p-a_0\leqq n-p$. Let $\{\lambda\}$ be the corresponding Schur function. To obtain the product of two Schubert symbols one considers the products $\{\lambda\}\{\mu\}$ of the corresponding Schur functions. This may be expressed as the sum of Schur functions $\{\nu\}$. The Schubert symbols which

appear in the product are those which correspond to Schur functions $\{\nu\}$ appearing in $\{\lambda\}\{\mu\}$, each with its appropriate numerical coefficient, and subject to the conditions (i) that the corresponding partition has not more than $p+1$ non-zero elements and (ii) that $\nu_0 \leqq n-p$. Applications are given.
J. A. Todd (Cambridge, England).

Chisini, Oscar. Sui teoremi d'esistenza delle funzioni algebriche di una e di due variabili. Rend. Sem. Mat. Fis. Milano **16**, 182–199 (1942).

The author presents an expository discussion of existence theorems of algebraic curves and surfaces with given ramifications. The methods described are those of the Italian school. *O. F. G. Schilling* (Chicago, Ill.).

Godeaux, Lucien. Sur quelques courbes planes contenant des involutions cycliques. Nieuw Tijdschr. Wiskunde **35**, 163–165 (1947).

v. Szökefalvi Nagy, Gyula (Julius). Reduzible algebraische Kurven vom Maximalindex in den mehrdimensionalen Räumen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 33–48 (1941). (Hungarian. German summary)

v. Szökefalvi Nagy, Gyula (Julius). Irreduzible algebraische Kurven vom Maximalindex in den mehrdimensionalen Räumen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 49–63 (1941). (Hungarian. German summary)

Cf. J. Reine Angew. Math. **186**, 30–39, 40–48 (1944); these Rev. **6**, 217.

Differential Geometry

de Mira Fernandes, Aureliano. Aspects of modern differential geometry. Revista Acad. Ci. Madrid **37**, 267–281 (1943). (Spanish)

Bouligand, Georges. Surfaces douées d'une famille d'asymptotiques ordinaires. Bull. Soc. Math. France **74**, 31–41 (1946).

Let $y(x,\lambda)$, $z(x,\lambda)$ be functions for which the derivatives y_λ, z_λ, $y_{\lambda x}$, $z_{\lambda x}$, y_{xx} and z_{xx} are continuous and $z_\lambda \neq 0$. Assume, moreover, that the lines $\lambda=$constant are asymptotic lines on the surface S: $y=y(x,\lambda)$, $z=z(x,\lambda)$; then $y_\lambda z_{xx}-z_\lambda y_{xx}=0$. The second family of asymptotic lines is in the classical case determined by a differential equation which involves $y_{\lambda\lambda}$ and $z_{\lambda\lambda}$. The present paper establishes without hypothesis regarding $y_{\lambda\lambda}$ and $z_{\lambda\lambda}$ the existence of asymptotic lines of the second family in a generalized sense. The lines are limits of ordinary asymptotic lines of regular surfaces S_ν which tend to S, and are also in a generalized sense perpendicular to their spherical images. *H. Busemann.*

Backes, F. Sur les congruences de cercles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 442–452 (1947).

Soient Γ, Γ' deux cercles de l'espace, dépendant de 2 paramètres u et v, situés sur une même sphère variable $S_5(u,v)$ du pentasphère mobile de référence, tels que chacun de ces cercles admette 2 sphères focales dans la congruence qu'il engendre, et qu'en outre les périsphères (enveloppes de sphères à un paramètre) de ces deux congruences se correspondent, avec les mêmes paramètres u et v. Alors, le cercle γ, dont les foyers sont les points A et B communs à Γ et Γ', admet 2 sphères focales, et les périsphères de sa congruence sont également données par $u=$constante, $v=$constante.

Inversement, soit γ un cercle possédant 2 sphères focales, u et v désignant les paramètres des périsphères de la congruence qu'il engendre. Il existe alors une infinité de cercles Γ, passant par les foyers AB de γ, situés sur la sphère $S_5(u,v)$ dont AB sont les points caractéristiques, admettant des sphères focales, et tels que u, v soient encore les paramètres des périsphères de la congruence engendrée par Γ. Chaque cercle Γ correspond à une solution ϕ de l'équation de Laplace à laquelle satisfont les coordonnées (par rapport à un pentasphère fixe) de la sphère $S_5(u,v)$, rapportée aux lignes principales de son enveloppe. *P. Belgodère* (Paris).

Backes, F. Sur un système cyclique particulier. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 563–566 (1947).

Consider the osculating circles C_u and C_v of the lines of curvature $(M)_u$ and $(M)_v$ which meet in a point M of a surface of Euclidean space. The circle Γ orthogonal to the surface and intersecting a second time each of the circles C_u and C_v is the intersection of the spheres of geodesic curvature of the lines of curvature $(M)_u$ and $(M)_v$. Of course the sphere of geodesic curvature at a point M of a curve on a surface is the sphere which passes through M and has its center at the center of geodesic curvature. The author determines all the surfaces for which the circles form a cyclic system. Aside from the trivial case of a sphere, the linear element of such a surface is reducible to the form $ds^2=(du^2+dv^2)/(U+V)^2$, where $U=U(u)$ and $V=V(v)$. The parametric curves are the lines of curvature and hence the surfaces are isothermic of special form. The axes of the circles Γ of any such surface describe a congruence whose focal surface degenerates into two curves. *J. DeCicco.*

Charrueau, André. Sur des congruences de droites ou de courbes déduites d'une surface quelconque. C. R. Acad. Sci. Paris **225**, 1055–1058 (1947).

En partant de la transformation $\tau(O_1, O_2, \mathbf{i}_1, \mathbf{i}_2)$ de contact de la seconde classe, qui est déterminée par deux points O_1, O_2 et deux vecteurs $\mathbf{i}_1$, $\mathbf{i}_2$ et qui a été définie par l'auteur [C. R. Acad. Sci. Paris **225**, 792–794 (1947); ces Rev. **9**, 200], l'auteur déduit les propriétés des éléments correspondants par τ et par la transformation τ^{-1}, inverse de τ. A un élément de contact la transformation τ^{-1} fait correspondre en général deux éléments de contact. L'auteur examine le cas où $\mathbf{i}_1$, $\mathbf{i}_2$ sont fixes et liés à O et aussi le cas où O_1 (O_2) varie sa position en fonction de t (u) de manière que τ reste invariable. Les cas où τ^{-1} fait correspondre à un élément quelconque en seul élément de contact et où $O\equiv O_1$, et aussi où $O_1\equiv O_2$, sont examinés. *F. Vyčichlo* (Prague).

Mishra, Ratan Shanker. Some properties of rectilinear congruences obtained by tensor method. Proc. Benares Math. Soc. (N.S.) **7**, no. 2, 41–49 (1945).

The author applies some tensor methods, involving dual numbers, which were developed by M. M. Slotnik [Math. Z. **28**, 107–115 (1928)] to the study of rectilinear congruences. The technique is interesting. Most of the results are well-known properties of rectilinear congruences.
N. Coburn (Ann Arbor, Mich.).

Mishra, Ratan Shankar. The five families of ruled surfaces through a line of rectilinear-congruence. J. Indian Math. Soc. (N.S.) **10**, 68–72 (1946).

The author employs the operation of forming the Jacobian [used by K. Ogura, T. Hayashi and R. Behari] in order to verify some of the equations obtained in the paper reviewed above. In addition a characterization of a Bianchi congruence is given. *N. Coburn* (Ann Arbor, Mich.).

van der Kulk, W. On line congruences. Proc. Nat. Acad. Sci. U. S. A. **34**, 9–12 (1948).

This paper states without proof some results concerned with the following problem. In a projective space of three dimensions, let there be given a two-parameter family of curves C, one curve C through each point of the space. Let Σ_p be a p-parameter family of surfaces, such that each surface of Σ_p has but one point in common with each curve C. In the one-to-one correspondence so established between pairs of surfaces of Σ_p, it is assumed that the asymptotic curves correspond. The maximal value of p is desired, and the kinds of curves C which admit such p-parameter families of transversal surfaces with maximal value of p. It is stated that the maximal value of p cannot exceed four, and the curves C which admit a four-parameter family of surfaces Σ_4 are necessarily straight lines. The developables of the congruence intersect each surface of Σ_4 in conjugate nets. The discussion is broken up into five cases according as the congruence of lines C is (a) nonparabolic, one of its focal surfaces not being degenerate; (b) parabolic with nondegenerate focal surface; (c) nonparabolic, both focal surfaces being curves; (d) parabolic with a curve for a focal surface; (e) composed of lines through a point. The results in each of these cases are given. In particular, in case (a) the congruence has a maximal value of $p=4$ if and only if its developables intersect both focal surfaces in R-nets. Such a congruence depends on six arbitrary functions of one variable. The family Σ_4 is uniquely determined by the congruence, and the asymptotics on each surface of Σ_4 and the focal surfaces correspond. The developables of the congruence intersect each surface of Σ_4 in R-nets. *V. G. Grove.*

Feld, J. M. A kinematic characterization of series of lineal elements in the plane and of their differential invariants under the group of whirl-similitudes and some of its subgroups. Amer. J. Math. **70**, 129–138 (1948).

Kasner began the study of the differential geometry of series and fields of lineal-elements [same J. **33**, 193–202 (1911)]. This subject was developed further in papers by Kasner and DeCicco [Kasner and DeCicco, same J. **59**, 545–563 (1937); **61**, 131–142 (1939); DeCicco, Trans. Amer. Math. Soc. **46**, 348–361 (1939); **47**, 207–229 (1940); these Rev. **1**, 84, 170]. In this study, the three-parameter group of whirls generated by turns and slides is important. The finite and differential invariants of the six-parameter group G_6, termed the whirl-motion group, were obtained in the preceding papers. The author considers the geometry of the seven-parameter whirl-similitude group G_7, generated by whirls, motions, and magnifications.

Under the group G_7, the author obtains the curvature κ and torsion τ of a series of lineal-elements. The intrinsic equations of a series within G_7 are $\kappa=f(\theta)$ and $\tau=g(\theta)$, where θ is the angle between fixed and variable lineal-elements of the series. New proofs are given of the corresponding results, first obtained by the reviewer, for G_6. The machinery of the complex variable is used. The author also prefers to regard a series of lineal-elements as a description of the ∞^1 positions that a plane takes when it is subjected to a continuous displacement over another plane. *J. DeCicco* (Chicago, Ill.).

Terracini, Alessandro. Sulla geometria delle equazioni differenziali. Ann. Mat. Pura Appl. (4) **25**, 277–286 (1946).

Exposition of the results of a series of recent papers by the author.

Petrov, P. I. Differential invariants of Riemann spaces. Doklady Akad. Nauk SSSR (N.S.) **58**, 1273–1274 (1947). (Russian)

A set of $6+5+7$ rational invariants is given, which form the simplest base of a complete system of scalar metrical differential invariants of the third order of a three-dimensional Riemannian space. Application is made to the case of an ordinary conformal Euclidean space. *D. J. Struik.*

Wu, George. Study of a surface by means of certain associate ruled surfaces in affine space. Amer. J. Math. **69**, 801–814 (1947).

Let us introduce the following notations: σ is a surface in affine 3-space, C a curve of σ, T_a $(a=1, 2)$ the asymptotic tangent of the ath system, R_a the geometric locus of T_a along C and $L_a(P, C)$ the Lie quadric of R_a at the generator point P of C [the Lie quadric of σ at P is obviously $L_1(P, T_2)=L_2(P, T_1)$], ω the osculating plane of C at P and finally K_a the intersection $[\omega, L_a(P, C)]$. If C varies but remains tangent to a fixed direction $T(P)$ at P, then the geometrical locus of K_a is a quadric $Q_a(T)$. In particular, $Q_1(T_2)=Q_2(T_1)=Q$ is the so-called canonical quadric of σ at P. The center of Q is a fixed point if and only if σ is a ruled affine sphere. Let t be an arbitrary tangent direction to σ at P and let $M_a(t)$ be the Moutard quadric of R_a, belonging to t. The residual conic of $M_1(t)$ and $M_2(t)$ is tangent to t if and only if t is a tangent of Segre. Some other theorems of this kind are derived. The paper is closely connected with the papers of B. Su [Tôhoku Math. J. **33**, 26–38 (1930); 190–198 (1931); Jap. J. Math. **9**, 233–238 (1933)]. [There are seriously confusing misprints on p. 803.] *V. Hlavatý* (Prague).

Varga, O. Aufbau der Finslerschen Geometrie mit Hilfe einer oskulierenden Minkowskischen Massbestimmung. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 14–22 (1942). (Hungarian. German summary)

Die Tatsache, dass der Finslersche Raum sich in der Umgebung einer Stelle wie ein Minkowskischer verhält, wird in der vorliegenden Arbeit zum Aufbau der Finslerschen Geometrie verwendet. Der Grundgedanken ist folgender: Das Minkowskische Raumelement, dass in einer gewissen Umgebung ein Finslersches Raumelement approximiert, wird auf ein solches Koordinatensystem bezogen, dass dadurch die Minkowskische Massbestimmung die Finslersche oskuliert. Diese oskulierende Massbestimmung ermöglicht es neben der Raummetrik auch den Euklidischen Zusammenhang des Raumes herzustellen. *Author's summary.*

Su, Buchin. On the isomorphic transformations of K-spreads in a Douglas space. Acad. Sinica Science Record **2**, 11–19 (1947).

The isomorphic transformations of K-spreads have already been considered by the reviewer [J. London Math. Soc. **18**, 100–107 (1943); these Rev. **5**, 152] and by the

author [Trans. Amer. Math. Soc. 61, 495–507 (1947); these Rev. 8, 602]. In those papers the equations defining the transformations are functions of position only. In this paper the author considers transformations which depend both upon position and upon the direction of the K-spreads. The conditions to be imposed upon the transformations in order that every K-spread of the space shall be carried into a K-spread are expressed by a set of equations similar to those given for functions of position only. The author shows that the absence of the operator "Lie derivation" prevents the effective examination of the integrability conditions.

E. T. Davies (Southampton).

Freidina, M. G. Dual systems admitting a group of motions. Doklady Akad. Nauk SSSR (N.S.) 57, 547–550 (1947). (Russian)

A double system consists of linear elements (x^1, x^2, x^3) (in Cartan's sense) at the basis of which there are two metric forms $ds_1=\overset{1}{\omega}_i dx^i$, $ds_2=\overset{2}{\omega}_i dx^i$ for the length of a "curve" $\overset{0}{\omega}_i dx^i=0$. The characteristic property of such a double system is the fact that the angular metric of one is conformal to that of the other. If $d\theta_1=\sqrt{\underset{1}{K}}ds_1$ and $d\theta_2=\sqrt{\underset{2}{K}}ds_2$, the functions $\underset{1}{K}(x^i)$, $\underset{2}{K}(x^i)$ are constant along null lines; this gives $\underset{1}{X}\underset{2}{K}=\underset{2}{X}\underset{1}{K}=0$, where $\underset{i}{X}=\xi_i^j\partial/\partial x_j$, ξ_i^j being the inverse matrix of $\overset{i}{\omega}_j$. The author obtains the "conditions of structure" on the ω's in order that they should define a double system. If the space (x^1, x^2, x^3) is mapped on the Euclidean plane $(x, y, z=dy/dx)$ so that the null lines of one metric correspond to points, then this metric takes on a Finsler form and in particular ds^2 becomes Gaussian if one of the curvatures $\underset{i}{K}$ is a constant. The author's aim is to construct such a double system with $\underset{1}{K}, \underset{2}{K}\neq$constant, which she does by further requiring that this space admit a one-parameter group of motions. She finds by integrating the conditions of structure that

$$ds_1=\frac{da}{\sqrt{A(a)}\sqrt{a+b}},\quad ds_2=\frac{db}{\sqrt{B(b)}\sqrt{a+b}},$$

where $A'=-4\underset{1}{K}$, $B'=-4\underset{2}{K}$, while the "curve" is given by

$$\left(\int_{b_0}^{b}\frac{db}{(AB)^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}\right)da-dx^3=0.$$

M. S. Knebelman (Pullman, Wash.).

Lee, Hwa-Chung. On skew-metric spaces and function groups. Amer. J. Math. 69, 790–800 (1947).

This is an extension of some results obtained in two previous papers [same J. 65, 433–438 (1943); 67, 321–328 (1945); these Rev. 5, 15; 7, 81] about spaces with a skew symmetric fundamental tensor $a_{\alpha\beta}$. The tensor $K_{\gamma\alpha\beta}=3\partial_{[\gamma}a_{\alpha\beta]}$ is called the curvature tensor of the space L_m. For a flat space ($K_{\gamma\alpha\beta}=0$) there exists a coordinate system for which $a_{\alpha\beta}$ has constant components. For spaces (L^m) with a contravariant fundamental tensor $a^{\alpha\beta}$ the curvature tensor is defined as $K^{\alpha\beta\gamma}=3a^{\delta[\alpha}\partial_\delta a^{\beta\gamma]}$. The vanishing of this tensor is a necessary and sufficient condition for the existence of coordinate systems for which $a^{\beta\gamma}$ is constant (flat space). A group space L^r is determined by r independent functions y^λ for which the "Poisson brackets" $b^{\lambda\mu}=(y^\lambda, y^\mu)$ are functions of the y^λ. The tensor of L^r is $b^{\lambda\mu}$. The L^r turns out to be flat if L^m is flat. The totality of functions of y^λ is called a function group in L^m. It is shown that in a flat space L^m every function group determines a "reciprocal group" consisting of the functions in involution with the functions of the given group.

J. Haantjes (Amsterdam).

Inglada García-Serrano, Vicente. Scalars, vectors and tensors. Revista Acad. Ci. Madrid 37, 282–293 (1943). (Spanish)

NUMERICAL AND GRAPHICAL METHODS

Alt, Franz L. A Bell Telephone Laboratories' computing machine. I. Math. Tables and Other Aids to Computation 3, 1–13 (1948).

The recent interest in digital computing machines has led to developments in two distinct directions: electronic and electro-mechanical. The author is concerned with one of the large scale machines of the latter type. This is the first of two articles describing an automatically sequenced electro-mechanical computing instrument recently built by the Bell Telephone Laboratories. The author gives a succinct description of the overall characteristics of the machine without attempting to be encyclopedic. He discusses the so-called "bi-quinary" system of representing decimal numbers as well as several alternative systems and indicates how they can be realized by two-position relays. He next describes the arithmetic organ of the machine and the so-called "floating decimal point." The paper closes with an explanation of the input-output organs of the machine. The instrument has an "inner" memory, in registers consisting of banks of relays, for 30 numbers each of seven decimal digits together with a characteristic from -19 to $+19$. In addition the teletype tapes of the machine are used for an additional "outer" memory as well as for the memory for instructions. It takes about 2 seconds to read an order or a number from the tapes into the machine and about 1 second to carry out a multiplication.

H. H. Goldstine (Princeton, N. J.).

***Tables of the Bessel Functions of the First Kind of Orders Sixteen Through Twenty-Seven.** By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1948. xi+764 pp. $10.00.

[For reviews of previous volumes cf. these Rev. 8, 406, 605; 9, 208.] The present volume carries on the work for $n=16$ to 27. In each case x runs from 0 to 100 at intervals of 0.01 and the functional values are given to 10 decimals.

A. Erdélyi (Pasadena, Calif.).

***Morgan, Samuel P. Tables of Bessel Functions of Imaginary Order and Imaginary Argument.** California Institute of Technology Bookstore, Pasadena 4, Calif., 1947. v+61 pp. $2.75.

In these tables the following functions are tabulated:

$$F_\nu(v)\equiv\tfrac{1}{2}\pi\frac{I_{i\nu}(v)+I_{-i\nu}(v)}{\operatorname{sh}\nu\pi}=\frac{\pi}{\operatorname{sh}\nu\pi}\Re I_{i\nu}(v),$$

$$G_\nu(v)\equiv\tfrac{1}{2}i\pi\frac{I_{i\nu}(v)-I_{-i\nu}(v)}{\operatorname{sh}\nu\pi}=-\frac{\pi}{\operatorname{sh}\nu\pi}\Im I_{i\nu}(v)$$

$$=K_{i\nu}(v)=\tfrac{1}{2}i\pi e^{-\nu\pi/2}H_{i\nu}^{(1)}(iv),$$

where ν is real and v is real and positive; $I_{i\nu}(v)=e^{\nu\pi/2}J_{i\nu}(iv)$. The functions $F_\nu(v)$ and $G_\nu(v)$ are called "wedge functions" of the first and second kinds, respectively, since in potential theory they show a certain analogy to the solutions of Legendre's equation called "cone functions." In these tables the argument $v=e^x$. The functions $F_\nu(e^x)$ and $G_\nu(e^x)$ were computed by step-by-step numerical integration of the differential equation $w''+(\nu^2-e^{2x})w=0$ on punched card machines, using a method of Feinstein and Schwarzschild [Rev. Sci. Instruments 12, 405–408 (1941); these Rev. 3, 156]. The tabular interval is 0.01 in x and 0.2 in ν. The function $F_\nu(e^x)$ is tabulated over the complete ranges $0.2\leqq\nu\leqq 10.0$; $-0.49\leqq x\leqq 2.50$. The function $G_\nu(e^x)$ is tabulated in the range $0.2\leqq\nu\leqq 10$. The range of x increases from $-0.49\leqq x\leqq +0.50$ to $0.49\leqq x\leqq 2.50$. The error in the last figure in any tabulated value does not exceed 5 units. In a separate table the values of $G_\nu(e^x)$, computed from $G_\nu(e^x)=\int_0^\infty e^{-e^x \operatorname{ch} t}\cos \nu t\, dt$, and correct to the last printed figure, are given for $x=1.00$, 1.50, 2.00 and 2.50 for those values of ν not included in the main table.

S. C. van Veen (Delft).

Goheen, H. E. A bound for the error in computing the Bessel functions of the first kind by recurrence. Bull. Amer. Math. Soc. 53, 972–975 (1947).

Let $e_n(x)$ represent the difference between the true value of $J_n(x)$ and the approximate value obtained by applying the recurrence relation $J_{n+1}(x)=2nx^{-1}J_n(x)-J_{n-1}(x)$ to the tabulated values of $J_0(x)$ and $J_1(x)$ (with errors $e_0(x)$ and $e_1(x)$). Then $e_n(x)$ is obtained from

$$\begin{vmatrix} e_0(x) & e_1(x) & e_n(x)\\ J_0(x) & J_1(x) & J_n(x)\\ Y_0(x) & Y_1(x) & Y_n(x)\end{vmatrix}=0.$$

By putting

$$E_n(x)=\tfrac{1}{2}\pi x Y_n(x)[J_1(x)+J_0(x)]-\tfrac{1}{2}\pi x J_n(x)[Y_0(x)+Y_1(x)]$$

or

$$\begin{vmatrix} 1, & -1, & E_n(x)\\ J_0(x), & J_1(x), & J_n(x)\\ Y_0(x), & Y_1(x), & Y_n(x)\end{vmatrix}=0$$

the following bound is obtained for all $x\geqq x_0>0$:

$$|e_n(x)|\leqq\left[\left\{\frac{|E_n(x_0)|+(\pi x_0+3)|J_n(x_0)|}{|J_1(x_0)+J_0(x_0)|}+\tfrac{1}{2}\pi(x_0+x)+2\right\}\times\{|J_0(x)|+|J_1(x)|\}+(\pi x+3)|J_n(x)|\right]\max(|e_0|,|e_1|).$$

This bound will be almost exact for $x=x_0$.

S. C. van Veen (Delft).

Ferguson, D. F., and Wrench, John W., Jr. A new approximation to π. II. Math. Tables and Other Aids to Computation 3, 18–19 (1948).

As a result of an independent calculation by Ferguson, 12 erroneous digits have been discovered in the previously published value of π [same journal 2, 245–248 (1947); these Rev. 8, 534]. The last 88 digits of this 808 digit value are therefore reproduced as amended. The value of arccot 5 is likewise corrected. *D. H. Lehmer* (Berkeley, Calif.).

Salzer, Herbert E. Coefficients for expressing the first twenty-four powers in terms of the Legendre polynomials. Math. Tables and Other Aids to Computation 3, 16–18 (1948).

"The . . . table of coefficients gives the exact expression for x^n, $n=0(1)24$, in terms of Legendre polynomials $P_m(x)$. . . . This extends the previous short tables (inadequate for many needs). . . . These coefficients are useful in obtaining an approximation for polynomials of high degree, which is best in a well-known least-square sense. . . . For methods of using these coefficients, one might consult the very instructive article by C. Lanczos . . ." [J. Math. Phys. Mass. Inst. Tech. 17, 123–199 (1938)]. *A. Erdélyi.*

Hoel, P. G., and Wall, D. D. The accuracy of the root-squaring method for solving equations. J. Math. Phys. Mass. Inst. Tech. 26, 156–164 (1947).

The authors emphasize that the transformed equations in Graeffe's method can be calculated by the use of punched cards. After this they give estimates of the power error and the rounding error. First they give limits for the moduli of the (real or complex) roots as calculated from the sth transformed equation. This formula is applied for determining the rate of error decrease when determining the roots x_i by successive transformed equations. Their result agrees with the reviewer's [Quart. Appl. Math. 4, 177–190 (1946), in particular, p. 182; these Rev. 8, 53] but their conclusions are different. In the second part of the paper they consider the rounding errors. *E. Bodewig* (The Hague).

Cassina, Ugo. Sulla risoluzione numerica delle equazioni e dei sistemi di equazioni algebriche o trascendenti. Rend. Sem. Mat. Fis. Milano 16, 156–181 (1942).

The paper gives a brief elementary description of the methods for solving a system of equations in one or more unknowns. *E. Bodewig* (The Hague).

Kantorovič, L. V. On the method of steepest descent. Doklady Akad. Nauk SSSR (N.S.) 56, 233–236 (1947). (Russian)

This concerns the solution by successive approximations of the equation (1) $Ax-\phi=0$, where A is a self-adjoint linear operator in real Hilbert space and ϕ is a given vector and x an unknown vector in that space. Particular attention is paid to speed of convergence and to practical applications for the numerical solution of integral and other functional equations. Reference is made to an earlier paper by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 455–460 (1945); these Rev. 8, 30]; the author acknowledges that he was unaware that his work had been anticipated in part by that of various British and American authors.

The method of steepest descents for the equation (1) determines the successive approximations by the following process. Let (2) $H(x)=(Ax,x)-2(\phi,x)$. Then if the form (Ax,x) is positive definite, H has an absolute minimum at a solution of (1). Let x_n be an approximation to a solution; the next approximation is then $x_{n+1}=x_n+t_nu_n$, where (a) u_n is a unit vector such that

$$\delta H=\left[\frac{d}{dt}H(x_n+t_nu_n)\right]_{t=0}$$

has its least possible value (i.e., u_n is the direction in which H descends most rapidly at x_n), and (b) $g(t)=H(x_n+tu_n)$ has a minimum at $t=t_n$. The convergence to a minimum of H for an H of very general form, including (2) as a special case, follows by the argument of Curry [Quart. Appl. Math. 2, 258–261 (1944); these Rev. 6, 52; see especially the remark at the end of § 1; but nothing is said there about speed of convergence]. For the case (2) the formula for x_{n+1}

becomes

$$x_{n+1}=x_n-\frac{(z_n, z_n)}{(Az_n, z_n)}\cdot z_n, \tag{3}$$

where $z_n=Ax_n-\phi$.

The results asserted by the author are now as follows. (I) If the spectrum of A is in the interval (m, M), where $0<m<M<\infty$, then the process converges so that

$$(x_n-x^*)^2\leqq m^{-1}\left(\frac{M-m}{M+m}\right)^{2n}(Ax_0-\phi, x_0-x^*),$$

where x^* is the root of (1).

(II) Under the same assumptions, except that $m=0$ and a root of (1) exists, then (a) if 0 is an isolated eigenvalue, the sequence (3) converges to some root of (1), and (b) in the general case we have $(Ax_n-\phi, x_n-x^*)=O(1/n)$.

(III) If A is an unbounded operator, one gets similar results (not stated in detail) provided one replaces (x, x) by (Bx, x) where B is a self-adjoint operator such that $(Bx, x)>(x, x)$ and where B^{-1} is defined in the whole space. The condition on A is $m\cdot(Bx, x)\leqq(Ax, x)\leqq M\cdot(Bx, x)$, where $0<m<M<\infty$. The successive approximations become

$$x_{n+1}=x_n-\frac{(Bx_n, x_n)}{(Ax_n, x_n)}\cdot z_n,$$

where $z_n=B^{-1}(Ax_n-\phi)$. Using results of Friedrichs [Math. Ann. 109, 465–487, 685–713 (1934)], this can be extended to certain cases where B is defined on a dense set Ω_0 and extended to a set Ω_B which is the closure of Ω_0 with respect to the metric (Bx, x). Then z_n is that element of Ω_B such that $(Bu, z_n)=(Au, x_n)-(u, \phi)$ for all $u\epsilon\Omega_0$. This is applied to the case where $Av=(av_x)_x+(bv_y)_y-cv$, $Bv=v_{xx}+v_{yy}$.

(IV) The convergence can be accelerated by making use of the expansion $x_{p+1}=x_0+\alpha_0 z_0+\alpha_1 A z_0+\cdots+\alpha_p A^p z_0$; one uses undetermined coefficients and determines the α_k so as to minimize H.

(V) If one takes for H, instead of (2), the form $H(x)=(Ax, x)/(x, x)$, then the minimum is the greatest lower bound of the eigenvalues. The method of steepest descents can then be applied. If A is positive definite and self-adjoint, and if the least eigenvalue λ_1 is isolated while λ_2 is the greatest lower bound of the remaining eigenvalues, then the convergence is as rapid as that of a geometric series with a ratio $\lambda_1^2/(2\lambda_2-\lambda_1)^2$. [No details are given. The statement requires correction, because if x_0 is an eigenvector corresponding to an eigenvalue $\lambda_3\geqq\lambda_2$, then, on a reasonable interpretation of the indeterminate situation, $x_n=x_0$ for all n, $H(x_n)=\lambda_3$.] *H. B. Curry* (State College, Pa.).

Garfath, H. L. Tchebycheff's mean value theorem and some results derivable therefrom. J. Inst. Actuaries Students' Soc. 7, 70–80 (1947).

The author derives numerous formulae of approximation for numerical integration and summation by replacing Tchebycheff's arguments by neighboring rational ones. For the same subject see Steffensen's paper [Skand. Aktuarietidskr. 28, 1–19 (1945); these Rev. 7, 219]. *E. Bodewig.*

Blume, H. Theorie und Praxis der Periodogrammanalyse von Registrierkurven, die im wesentlichen aus nichtpersistenten Wellenzügen bestehen. Z. Angew. Math. Mech. 25/27, 113–118 (1947). (German. Russian summary)

Die Arbeit behandelt die Berechnung der Frequenzen und Amplituden einer Kurve, die aus nichtpersistenten Wellenzügen besteht. Sie beschreibt die drei Fälle: lineare Amplitude, exponentielle Amplitude, beliebig veränderliche Amplitude. *E. Bodewig* (Den Haag).

Willers, Fr. A. Zur Bestimmung der mittleren Abscisse. Z. Angew. Math. Mech. 25/27, 29 (1947).

Zieht man zur Sehne eines Bogens eine parallele Tangente, legt durch die Mitte der Sehne eine y-Parallele und teilt das Stück zwischen Sehne und Tangente in 3 gleiche Teile, so gibt bekanntlich der der Tangente zuliegende Teilpunkt die mittlere Ordinate. Verfasser bestimmt nun den Fehler dieser Näherung: $\Delta=h^5[f_0^{\mathrm{iv}}/90-f_0'''f_0'''/(54f_0'')]+\cdots$, wo $2h$ die Intervallänge ist und die Mitte der Sehne die Abscisse $x=0$ hat. Die Konstruktion wird also z.B. unbrauchbar, wenn der Kurvenabschnitt einen Wendepunkt enthält. *E. Bodewig* (Den Haag).

Le Heux, J. W. N. The growth-curve. Nederl. Akad. Wetensch., Proc. 50, 1201–1213 = Indagationes Math. 9, 548–560 (1947).

A graphical method for fitting the logistic equation $N=Ae^{b(t-t_0)}/\{1+e^{b(t-t_0)}\}$. The method seems to have few advantages over the linearization given by the plot of $\log(A-N)/N$ against t. *C. P. Winsor* (Baltimore, Md.).

ASTRONOMY

Becq, G. La détermination des orbites elliptiques. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 4° (2) 13, no. 2, 117 pp. (1948).

In the common methods of determining an elliptic orbit from three observations an essential requirement is that the intervals between the observations should be a small fraction of the period of revolution of the observed object. In this paper a method is developed in which the interval between the first and third observations is arbitrary, up to about the period of revolution. The procedure is that of finding by successive approximations the definitive value of the semi-major axis and of the heliocentric arcs through which the planet has traveled during the intervals defined by the observations. All developments in powers of the time are avoided, and trigonometric expansions in terms of the eccentric anomaly are used. The calculations turn out to be laborious. Even though the author succeeds in making considerable abbreviations to the procedure as first developed, he concludes that his contribution is primarily of theoretical interest. It is further remarked that the method of Gauss may be considered to be a special case for small intervals of the method developed in this paper. *D. Brouwer* (New Haven, Conn.).

Silva, Giovanni. Sulla traiettoria dei satelliti rispetto al sole. Mem. Soc. Astr. Ital. (N.S.) 18, 113–132 (1947).

L'objet de ce travail est l'analyse de la trajectoire d'un satellite d'une planète par rapport au Soleil, en supposant que les mouvements composants soient kepleriens sur deux plans différents. L'auteur recherche des conditions qui assurent la concavité de la trajectoire et de sa projection sur le plan de l'orbite de la planète ou des cas dans lesquels la convexité peut intervenir à chaque révolution synodique du satellite ou enfin des conditions pour l'existence et non

existence d'un noeud à chauque révolution synodique de sorte que le mouvement devient périodiquement retrograde ou direct. L'auteur retrouve dans ces conditions plus générales la proposition connue que la Lune seulement parmi tous les satellites du système solaire a sa trajectoire toujours concave par rapport au Soleil et que seulement les deux satellites de Saturne (I, II) et les trois de Jupiter (V, I, II) les plus prochains aux planètes correspondantes doivent avoir un mouvement retrograde. *G. Lampariello.*

Silva, G. Moto dei satelliti intorno al Sole. I. Moti diretti o retrogradi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 228–233 (1947).

Silva, G. Moto dei satelliti attorno al Sole. II. Concavità o convessità della traiettoria verso il Sole. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 378–383 (1947).

Silva, G. Moto dei satelliti attorno al Sole. III. Pianeta e satellite con orbite circolari su piani comunque inclinati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 726–729 (1947).

In these papers the author continues his analysis of orbits of satellites relative to the sun [cf. the preceding review]. The orbits are classified on the basis of concavity into three classes. Necessary and sufficient conditions for the three types are formulated and established in special cases; the conditions are inequalities involving the constants of the elliptic orbits of the planet relative to the sun and of the satellite relative to the planet. *W. Kaplan.*

Clark, G. L. The decay of the gravitational energy of a double star. Monthly Notices Roy. Astr. Soc. **106**, 457–463 (1946).

The paper is a continuation of the author's previous investigations of the problem of loss of energy of a rotating cohesive system on the general theory of relativity [cf. Proc. Roy. Soc. London. Ser. A. **177**, 227–250 (1941); these Rev. **3**, 212]. The present investigation is concerned with the relativistic dynamics of a system of two spherical particles of equal mass, whose density is symmetrical about their respective centers, revolving in circular orbits around the common center of mass. It is shown that such a dynamical system, referred to as a double star, is characterized by a secular loss of the gravitational energy, and the magnitude of its decay is estimated to be of the order of $m^3b^4\omega^6$, where m denotes the mass of either particle, b is a quantity of the order of magnitude of linear dimensions of the double-star system and ω stands for the angular velocity of revolution. *Z. Kopal* (Cambridge, Mass.).

Irwin, John B. Tables facilitating the least-squares solution of an eclipsing binary light-curve. Astrophys. J. **106**, 380–426 (1947).

Sokolov, Yu. D. On a spatial homographic motion of a system of three material points. Doklady Akad. Nauk SSSR (N.S.) **58**, 369–371 (1947). (Russian)

The author studies the conditions for the homographic motion of three material particles under forces of mutual attraction. If the forces are meromorphic functions of the masses and distances $m_im_jf(\Delta_{ij})$ and the motion is not restricted to one plane then $f(\Delta)=A\Delta+B\Delta^{-3}$, $B\neq 0$.

J. Lifshits (Cambridge, Mass.).

Schwarzschild, M. On stellar rotation. II. Astrophys. J. **106**, 427–456 (1947).

[For part I cf. the same J. **95**, 441–453 (1942); these Rev. **3**, 281.] The paper gives a numerical integration of the equations of equilibrium of a slowly rotating star. The assumptions on which the present solution rests are that meridional currents in stars are negligible and that viscosity is negligible except in the central convective core. Arguments for these assumptions have been given earlier by the author. It is found that the solution shows an equatorial acceleration at the surface. This is in qualitative agreement with solar observations, but quantitatively the solution disagrees with the solar equatorial acceleration in giving a value about twice the observed one. The solution does not fulfill any condition imposed by viscous forces at the surface and it is assumed that the observed turbulence in the outer layer of the sun will produce only unimportant convection phenomena. *G. Randers* (Oslo).

Kourganoff, Vladimir. Sur la solution du problème des atmosphères modèles où le coefficient d'absorption est une fonction quelconque de la fréquence. C. R. Acad. Sci. Paris **225**, 1124–1126 (1947).

An approximate method is suggested for solving the equation of transfer appropriate for a stellar atmosphere in local thermodynamic equilibrium and with a constant net flux of radiation in all wavelengths. *S. Chandrasekhar.*

RELATIVITY

Schrödinger, Erwin. The final affine field laws. I. Proc. Roy. Irish Acad. Sect. A. **51**, 163–171 (1947).

In this paper, which is self contained, the author applies the method of his previous work [same vol., 41–50 (1946); these Rev. **8**, 412] to obtain a proposed set of unified field equations from the variational problem $\delta\int L d\tau=0$, where the integral is taken over a general four-dimensional manifold with an asymmetric affine connection which is varied. The scalar density L is given by $L=2\lambda^{-1}(-\det R_{\mu\nu})^{\frac{1}{2}}$ and $R_{\mu\nu}$ is the contracted curvature tensor formed from the affine connection. The field equations thus obtained are written in two different forms. *A. H. Taub.*

Schrödinger, Erwin. The final affine field laws. II. Proc. Roy. Irish Acad. Sect. A. **51**, 205–216 (1948).

In the first part of the paper the author classifies the field theories obtainable from the variational principle $\delta I=\delta\int \mathfrak{g}^{kl}R_{kl}d\tau=0$, where the R_{kl} is the Ricci tensor of a general affine connection Γ^i_{kl} and the contravariant density $\mathfrak{g}^{kl}$ is subject to the condition that its determinant is negative and serves to define a tensor $g^{kl}=\mathfrak{g}^{kl}/(-g)^{\frac{1}{2}}$, $g=\det \mathfrak{g}^{kl}$. The various field theories arise from (1) regarding only the g^{kl}, or only the Γ^i_{kl}, or both as independent variables, and (2) assuming that one or both of the entities g_{ik} and Γ^i_{kl} are symmetric. There are eight possibilities in all. The unified field theory put forth by the author in previous papers [cf. the preceding review] is one in which only the Γ^i_{kl} is varied independently and it is nonsymmetric.

The second part of this paper contains some remarks about the differential equations satisfied by the antisymmetric tensors involved in this theory. It is pointed out that these are not Maxwell's equations for empty space for the pair of anti-symmetric tensors which are supposed to

describe the electromagnetic field. It is also remarked that the theory proposed is symmetric in the sign of the proposed charge. *A. H. Taub* (Princeton, N. J.).

Schrödinger, Erwin. The final affine field laws. III. Proc. Roy. Irish Acad. Sect. A. **52**, 1–9 (1948).

[Cf. the preceding two reviews.] The author uses the fact that the field equations are derivable from a variational principle to deduce twenty-four identities between the derivatives of these field equations. The method used is to study the variation produced in the Lagrangian by an arbitrary infinitesimal transformation of coordinates. The discussion follows closely on that of Pauli [Encyklopädie Math. Wiss., vol. V 19, 1920, § 23]. *A. H. Taub.*

Schrödinger, Erwin. The relation between metric and affinity. Proc. Roy. Irish Acad. Sect. A. **51**, 147–150 (1947).

In §§ 1 and 2 of this paper the author recalls the result of L. P. Eisenhart [Non-Riemannian Geometry, Amer. Math. Soc. Colloquium Publ., v. 8, New York, 1927, p. 84] that Riemannian spaces are a subclass of spaces with symmetric connections for which there exists a homogeneous quadratic first integral of the equations of the paths. In § 3 the author considers field equations obtained from the variational principle $\delta\int(-g)^{\frac{1}{2}}g^{ik}R_{ik}d\tau=0$, where R_{ik} is the Ricci tensor formed from the metric tensor g_{ik} and the g_{ik} are varied. He shows that the same field equations are obtained from this variational principle if R_{ik} is considered as formed from a more general affine connection $\Gamma^i_{kl}=\{^{\,i}_{kl}\}+g^{is}T_{skl}$, $T_{skl}+T_{kls}+T_{lsk}=0$, and the g_{ik} and T_{skl} are varied. The remarks in § 4 are superseded by the first of the author's papers reviewed above. *A. H. Taub.*

Chang, T. S. Field theories with high derivatives. Proc. Cambridge Philos. Soc. **44**, 76–86 (1948).

Relativistic field theories are generalized to Lagrangians which involve higher derivatives of the field variables and corresponding generalized expressions are obtained for the current, energy-momentum and angular momentum tensors, and the symmetrized energy-momentum tensor. It is shown that these expressions may be made gauge-invariant by means of a subtraction procedure. The equations of motion are also expressed in Hamiltonian form in a gauge-invariant manner.

An alternative definition of the energy-momentum tensor is as $-2(-g)^{\frac{1}{2}}$ times the functional derivative of $\int L(x')\{-g(x')\}^{\frac{1}{2}}d^4x'$ with respect to g_{ij}. It is shown that at points at which space-time is flat this definition of the energy-momentum tensor differs from the corresponding symmetrized tensor obtained by the author by terms of zero divergence. *H. C. Corben* (Pittsburgh, Pa.).

Meksyn, D. Relativity of accelerated motion. Nature **160**, 834–835 (1947).

The author points out that transformations to accelerated coordinate systems are, in general, singular on certain surfaces. He considers the radiation attributed to a uniformly moving electron by an accelerated observer. *A. Schild.*

García, Juan. On the theory of space. Revista Acad. Ci. Madrid **36**, 263–295 (1942). (Spanish)

Jaiswal, J. P. On the electric potential of a single electron in gravitational fields. II. Proc. Benares Math. Soc. (N.S.) **7**, no. 2, 1–8 (1945).

This is a sequel to the author's earlier paper [same Proc. (N.S.) 7, no. 1, 17–25 (1945); these Rev. **8**, 175]. The electric potential in Einstein's general law of gravitation is obtained for the metric

$$ds^2=\left\{1-\frac{2m}{r}-\tfrac{1}{3}\alpha r^2\right\}dt^2$$
$$-\frac{1}{c^2}\left\{\frac{dr^2}{1-2m/r-\frac{1}{3}\alpha r^2}+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right\}.$$

The angular part of the potential depends again upon trigonometric and associated Legendre functions. The solution of the radial equation is given in power series.

C. Kikuchi (East Lansing, Mich.).

Ivanenko, D., and Sokolov, A. New consequences of the quantum theory of gravitation. Doklady Akad. Nauk SSSR (N.S.) **58**, 1633–1636 (1947). (Russian)

MECHANICS

Glagolev, A. A. On the construction of Burmester's points. Doklady Akad. Nauk SSSR (N.S.) **58**, 1881–1882 (1947). (Russian)

The connecting-rod of a four-bar linkage may be made to assume five arbitrary positions in the fixed plane. For these positions there exist four sets of five corresponding points, each set being located on a circle whose center is called a Burmester point. This paper gives a projective geometry construction for locating these points in a manner suggested by the analytic procedure used by Hackmüller [Z. Angew. Math. Mech. **18**, 252–254 (1938)].

M. Goldberg (Washington, D. C.).

Fadle, Johann. Eine einfache Konstruktion zur Auffindung des Trägheitsradius für eine beliebige Achse. Z. Angew. Math. Mech. **25/27**, 271–272 (1947).

Haag, Jules. Sur le pendule conique. C. R. Acad. Sci. Paris **225**, 1234–1236 (1947).

The author studies the suitability of the conical pendulum for use as the regulating element of a clock. He shows that the rate of the pendulum is highly sensitive to small variations of the friction in the driving mechanism. (Under realistic conditions the error of a clock regulated by such a pendulum might amount to as much as two seconds per day.) He also shows that the decay of departures from the steady state is very slow. It is concluded that the conical pendulum, considered as the regulating element of a clock, is decidedly inferior to the oscillating pendulum.

L. A. MacColl (New York, N. Y.).

Roth, E., und Sänger, R. Kritische Betrachtungen über die Verfahren von S. Dufrénois und O. von Eberhard zur Bestimmung der ballistischen Luftdichte. (Ballistische Störungstheorie.) Schweiz. Arch. Angew. Wiss. Tech. **14**, 22–32 (1948).

Chazy, Jean. Sur une généralisation des équations canoniques. C. R. Acad. Sci. Paris **226**, 19–23 (1948).

L'auteur propose une généralisation des équations d'Hamilton. Soit un système différentiel quelconque d'ordre $2n$

qu'on peut poser sous la forme suivante:

$$\dot{x}_i = Y_i, \quad \dot{y}_i = -X_i, \qquad i=1, \cdots, n. \tag{1}$$

Les deuxièmes membres dépendent de la variable t et des fonctions inconnues x_i, y_i deux à deux conjuguées. Si l'on remplace les variables x_i, y_i par des variables x_i', y_i' qui soient fonctions des variables x_i, y_i, t deux à deux conjuguées aussi et enfin telles que

$$\sum_i y_i dx_i - \sum_i y_i' dx_i' = d\varphi + \psi dt, \tag{2}$$

où φ et ψ désignent deux fonctions des variables x_i, y_i, t, le système transformé pour les inconnues x_i', y_i' est

$$\dot{x}_i' = Y_i', \quad \dot{y}_i' = -X_i', \qquad i=1, \cdots, n, \tag{1'}$$

la forme pfaffienne $\sum_i (X_i'\delta x_i' + Y_i'\delta y_i')$ étant la transformée de $\sum_i (X_i \delta x_i + Y_i \delta y_i) - \delta\psi$ où δ désigne une différentielle virtuelle.

Après avoir démontré cette proposition générale qui se réduit au théorème bien connu de la théorie des transformations canoniques (ou de contact) si en particulier les fonctions X_i, Y_i sont les dérivées partielles par rapport à x_i et y_i d'une fonction F, l'auteur en fait une application aux équations du mouvement d'un système de points matériels. Les points étant libres, les équations de Newton sont $m_i \ddot{x}_i = X_i$ $(i=1, \cdots, n)$, en désignant par $m_1 = m_2 = m_3$ la masse du point P_1 de coordonnées x_1, x_2, x_3, par $m_4 = m_5 = m_6$ la masse de $P_2(x_4, x_5, x_6)$ et ainsi de suite. Si l'on prend comme conjuguée de x_i la variable $y_i = m_i \dot{x}_i$, les équations de Newton s'écrivent

$$\dot{x}_i = y_i/m, \quad \dot{y}_i = -(-X_i), \qquad i=1, \cdots, n. \tag{I}$$

Alors, soit $2T$ la force vive du système. En remplaçant les x_i par des variables lagrangiennes q_i et en prenant comme conjuguée de q_i la variable $p_i = \partial T/\partial \dot{q}_i$, on voit que la condition (2) est satisfaite. Si l'on désigne par T_2, T_1, T_0 les groupes des termes de T de degrés 2, 1 et 0 par rapport aux vitesses lagrangiennes $\dot{q}_i$, par $\sum_i Q_i \delta q_i$ le travail virtuel des forces agissant sur les points du système, les équations transformées des équations (I) sont

$$\dot{q}_i = \partial T_2/\partial p_i, \quad \dot{p}_i = -\partial(T_2 - T_0)/\partial q_i + Q_i, \qquad i=1, \cdots, n. \tag{I'}$$

Un raisonnement de Poincaré permet d'étendre ces équations aux systèmes matériels comportant des liaisons sans frottement. Dans les équations (I'), les fonctions Q_i ne dérivent pas d'une fonction des variables q_i, t et peuvent dépendre encore des vitesses $\dot{q}_i$ et des variables p_i.

G. Lampariello (Messine).

Hydrodynamics, Aerodynamics, Acoustics

Kravtchenko, Julien. Sur l'existence des solutions du problème de représentation conforme de Helmholtz. Cas des arcs sans tangente. Ann. Sci. École Norm. Sup. (3) 63 (1946), 161–184 (1947).

The wake considered in this paper is due to the presence of an obstacle in a parallel horizontal channel. The obstacle is given by an arc $x = x(y)$, where $x(y)$ satisfies a Lipschitz condition. The theory of functional equations, due to Leray and Schauder, is not directly applicable to this case. However, the author obtains existence theorems by approximating the obstacle by smooth arcs, thus generalizing a procedure formerly employed by A. Weinstein [Math. Z. 31, 424–433 (1929)]. The essential tools are some classical theorems of Montel and Carathéodory [P. Montel, Leçons sur les Familles Normales des Fonctions Analytiques, Paris, 1927].

A. Weinstein (Washington, D. C.).

Groen, P. Internal waves in certain types of density distribution. Nature 161, 92 (1948).

The equation

$$\frac{d^2\varphi}{dz^2} + \left(\frac{g\,dS/dz}{c^2 S_0} - m^2\right)\varphi = 0, \quad \varphi(x, z, t) = \varphi(z) \exp i(mx - nt)$$

is proposed for the stream function φ of simple harmonic waves of small amplitude and speed $c = n/m$ in fluid under gravity. Here $S = S(z)$ is the specific volume at height z in the unperturbed state and S_0 is the mean specific volume. Some results are given without proof for the distribution $S = S_0 + \frac{1}{2}\Delta S \tanh(2z/b)$, where ΔS is the total variation of the specific volume and b is the thickness of the transition layer.

L. M. Milne-Thomson (Greenwich).

Guderley, Gottfried. Störungen in ebenen und rotationssymmetrischen Schall- und Überschallparallelstrahlen. Z. Angew. Math. Mech. **25/27**, 190–195 (1947). (German. Russian summary)

The problem investigated is the stationary disturbances of a two-dimensional or rotationally symmetric parallel uniform stream of sonic or supersonic velocity. The axis of symmetry is taken as the x-axis whose direction is also that of the undisturbed uniform velocity w_0. Using the new nondimensional characteristic variables ξ, η, with $\xi = 0$ representing the Mach wave and with the same origin as the (x, y) coordinate system, the disturbed velocity potential Φ for $\xi > 0$ can be written as $\Phi \cong w_0 l\{\xi \sin \alpha_0 + \eta \cos \alpha_0 + f(\eta)\xi^m\}$, where l is the representative length and α_0 is the Mach angle of the undisturbed flow. This gives a disturbance which is roughly of constant profile along the Mach line but is varying in strength. The function $f(\eta)$ is determined by the equations of motion. For supersonic velocities, disturbances exist only for $m \geqq 2$. For sonic velocity, disturbances exist only for $m \geqq 3$. Solutions for $f(\eta)$ are given for all these cases. The author shows that the result can be interpreted to mean the possibility of achieving an exactly uniform parallel sonic stream from subsonic velocities with a nozzle of finite length. The same is not true for achieving an exactly uniform parallel subsonic stream.

H. S. Tsien (Cambridge, Mass.).

Sibert, H. W. Approximations involved in the linear differential equation for compressible flow. J. Aeronaut. Sci. 14, 680–681 (1947).

By considering the order of magnitude of the different terms in the exact differential equation for the velocity potential of two-dimensional flows of compressible fluid, the author essentially shows that linearization of the differential equation is not possible for transonic speeds and for hypersonic speeds.

H. S. Tsien (Cambridge, Mass.).

Heaslet, Max. A., and Lomax, Harvard. The use of source-sink and doublet distributions extended to the solution of arbitrary boundary value problems in supersonic flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1515**, 48 pp. (1948).

It is desired to establish a method of constructing solutions to the linearized supersonic-flow potential equation (i.e., the wave equation) $\square^2\Omega = \Omega_{xx} - \Omega_{yy} - \Omega_{zz} = 0$ that is analogous to the usual source-sink distribution by which

solutions are constructed in incompressible potential flow. The latter is, mathematically, an application of Green's formula. This formula can be written, for use with the wave equation,

$$(1)\quad \iint_S \left(\sigma\frac{\partial\Omega}{\partial\nu}-\Omega\frac{\partial\sigma}{\partial\nu}\right)dS=\iiint_R (\Omega\Box^2\sigma-\sigma\Box^2\Omega)dR,$$

where ν denotes the co-normal to the surface S enclosing the region R. If the "supersonic source" potential

$$1/r_s=[(x-x_1)^2-(y-y_1)^2-(z-z_1)^2]^{-\frac{1}{2}}$$

is taken for σ, the analogy with the incompressible case is set up. But here the region R is bounded by certain Mach cones defining the region that influences conditions at any field point and, on one of these cones, $1/r_s$ becomes infinite. To pursue the source-sink analogy further in spite of this situation, the authors employ the method of Hadamard [Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, 1923; see also Courant and Hilbert, Methoden der mathematischen Physik, v. 2, Springer, Berlin, 1937, pp. 430–448]. Hadamard established the relation (1) for the finite parts of the integrals involved. In the source-sink analogy, the finite parts are found to vanish over both of the Mach cones and $\Omega(x, y, z)$ is finally determined by an integration over part of the wing surface.

The authors apply this theory to three problems treated elsewhere by other methods: (a) a symmetrical nonlifting airfoil, (b) a semi-infinite wedge with leading edge swept behind the foremost Mach cone, (c) the load distribution over a thin lifting surface. They then proceed to two new calculations: the load distributions for (i) a rolling wing whose leading edge lies ahead of the foremost Mach cone, and (ii) the same wing executing a pitching motion.

W. R. Sears (Ithaca, N. Y.).

Notes and tables for use in the analysis of supersonic flow. By the staff of the Ames 1- by 3-foot supersonic wind-tunnel section. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1428**, 73 pp. (10 plates) (1947).

This is a collection of formulae and data that have been found useful in high-speed wind-tunnel work. First, there appears a summary of fundamental formulae of thermodynamics and formulae relating to flow of a compressible fluid along stream tubes. There follows a section devoted to the differential equations of motion of a frictionless fluid, in various coordinate systems and including the linearized (small-perturbation) forms for steady motion. Succeeding sections present the formulae of supersonic nozzles, normal and oblique shock waves, and Prandtl-Meyer flow, i.e., isentropic plane flow about a corner. Finally, a rather long section is devoted to the results of supersonic airfoil theory. The "small-perturbation airfoil section theory" in this report includes both the Ackeret and Busemann approximations. A "large-deflection section theory" is also given, namely, the calculation using oblique shock waves and Prandtl-Meyer expansions, as is the modification of Ackeret's approximation for infinite yawed airfoils. A section entitled "Flow about wedges and cones" simply refers to the tables and to the well-known papers of Taylor and Maccoll [Proc. Roy. Soc. London. Sec. A. **139**, 278–311 (1933)] and Maccoll [ibid. **159**, 459–472 (1937)]. Certain data on viscosity of air, humidity, etc., are given in brief appendices.

Table I presents numerical results for subsonic isentropic channel flow; namely, pressure, density, temperature, speed of sound and area ratio as functions of Mach number. Table II has the same data for supersonic flow, and in addition the dynamic pressure, Mach angle and maximum Prandtl-Meyer angle. Table III gives pressure, density, temperature, total-head and speed-of-sound ratios for normal shock waves. In table IV are tabulated the functions $C_1\ (=2/(M_0^2-1)^{\frac{1}{2}})$ and C_2 of supersonic airfoil theory, as functions of the Mach number M_0. The properties of the N.A.C.A. tentative standard atmosphere are given in table V, up to an altitude of 260,000 ft.; these include temperature, speed of sound, pressure, density, and viscosity.

The reviewer notes that there is no mention throughout the report of the method of characteristics, which is commonly used in large-deflection airfoil calculations, accurate design of nozzles, flow over bodies of revolution and nonstationary problems. In fact, the absence of formulae for time-dependent processes is conspicuous. *W. R. Sears.*

★**Summaries of Foreign and Domestic Reports on Compressible Flow. Prepared by the Graduate Division of Applied Mathematics, Brown University, for the Analysis Division, Intelligence Department.** Technical Reports nos. **F-TR-1168A-ND, F-TR-1168B-ND, F-TR-1168C-ND, F-TR-1168D-ND.** Headquarters Air Materiel Command, Wright Field, Dayton, Ohio, 1947. Vol. I, vii+78 pp.; Vol. II, viii+100 pp.; Vol. III, vii+77 pp.; Vol. IV, viii+112 pp. Eng ?

Gunn, J. C. Linearized supersonic aerofoil theory. I, II. Philos. Trans. Roy. Soc. London. Ser. A. **240**, 327–373 (1947).

Let $U\phi$ denote the potential of the disturbance velocities u, v, w in a case of steady supersonic potential flow about a thin wing. Then the differential equation for ϕ, according to the linear-perturbation approximation, is (1) $\phi_{xx}+\phi_{yy}=\alpha^2\phi_{zz}$, where x, y, z are rectangular Cartesian coordinates, z in the direction of the uniform stream, whose velocity is U, and α^2 denotes M^2-1, M being the stream Mach number. In terms of Laplace transforms, this equation is written (2) $\bar{\phi}_{xx}+\bar{\phi}_{yy}=\alpha^2p^2\bar{\phi}$. It is first shown how this equation can be solved for a rectangular finite airfoil at zero incidence. The result is the same as would be written down immediately for a distribution of "supersonic sources and sinks." Other cases treated are the finite rectangular flat plate at incidence and an airfoil with curved leading edge, wing tips cut off parallel to the stream. In these cases the method employed is to utilize the Green's functions for (2) (in an (x, y)-plane) with the boundary consisting of the positive x-axis. These functions are found by a method attributed to Carslaw [Proc. London Math. Soc. (1) **30**, 121–161 (1899)] and Sommerfeld. The rectangular flat plate at incidence is treated in detail, including the case where the chord is very great and there is repeated overlapping of waves; numerical values are tabulated. In an attempt to treat a tapered (in thickness) rectangular airfoil the author appears to have inadvertently "tapered" its incidence as well, producing a case of doubtful physical interest.

In part II of the paper, the operational technique is extended to sweptback airfoils. First, airfoils at zero incidence are considered and the drag of constant-chord sweptback wings having double-wedge profiles is evaluated. The case of an airfoil at incidence, its edges not parallel to the stream, is illustrated by a certain triangular flat plate, one of whose leading edges lies outside the Mach cone of the nose and one inside. The Green's function method is used.

This technique is available for other cases, provided at least one edge lies outside the Mach cone, but the calculations appear to be lengthy. By superposition, the solution can be constructed for an airfoil with a straight (unswept) leading-edge portion and edges swept back behind the Mach cone on both sides. Finally, the "arrow" wing, i.e., a triangular wing whose leading edges are both behind the Mach cone, is produced as a limiting case when the middle portion vanishes.

Throughout this paper, use is made of physical analogies, e.g., the wave motion created on the surface of a body of water by the prescribed motion of certain finite plane vertical barriers. *W. R. Sears* (Ithaca, N. Y.).

Snow, Robert M. Aerodynamics of thin quadrilateral wings at supersonic speeds. Quart. Appl. Math. **5**, 417–428 (1948).

Germain, Paul. Application de la composition des mouvements coniques au calcul aérodynamique de l'aile rectangulaire en régime supersonique. C. R. Acad. Sci. Paris **226**, 311–313 (1948).

Hantzsche, W., and Wendt, H. Zum Kompressibilitätseinfluss bei der laminaren Grenzschicht der ebenen Platte. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I517–I521 (1940).

The essential content of this paper is contained in the more general later paper of the same authors [see the following review]. The case of the insulated plate for $\sigma=1$ and $\omega=0.5$, 0.8 and 1.0 and for arbitrary σ with $\omega=1$ is discussed here. *H. W. Liepmann* (Pasadena, Calif.).

Hantzsche, W., und Wendt, H. Die laminare Grenzschicht der ebenen Platte mit und ohne Wärmeübergang unter Berücksichtigung der Kompressibilität. Jahrbuch **1942** der Deutschen Luftfahrtforschung, I40–I50 (1942).

The authors transform the equations for the laminar boundary layer on a flat plate into the following system [see also the following review]:

$$\{-\tfrac{1}{2}\rho u\chi+\rho\bar{v}\}\frac{du}{d\chi}=\frac{d}{d\chi}\left\{\mu\frac{du}{d\chi}\right\},$$

$$-\tfrac{1}{2}\chi d\rho u/d\chi+d\rho\bar{v}/d\chi=0,$$

$$\{-\tfrac{1}{2}\rho u\chi+\rho\bar{v}\}\frac{di}{d\chi}=\frac{d}{d\chi}\left[\frac{\lambda}{c_p}\frac{di}{d\chi}\right]+\mu\left(\frac{du}{d\chi}\right)^2,$$

with $\chi=yx^{-\frac{1}{2}}$; writing $G(u)=\mu du/d\chi$ and $\sigma=c_p\mu/\lambda$ for the Prandtl number one obtains $G_ui_u=\sigma^{-1}(G_ui_u+Gi_{uu})+G$, $GG_{uu}=-\frac{1}{2}\rho\mu u$, or in dimensionless variables defined by $\eta=u/U$, $G(u)=(\frac{1}{2}\rho_0\mu_0U^3)^{\frac{1}{2}}g(\eta)$, $i(u)=U^2j(\eta)$:

$$(*)\qquad\begin{cases}gg_{\eta\eta}=-\eta(j_0/j)^{1-\omega},\\(1-\sigma)g_\eta j_\eta+g(j_{\eta\eta}+\sigma)=0,\end{cases}$$

where $\mu/\mu_0=(T/T_0)^\omega=(j/j_0)^\omega$, $\rho/\rho_0=T_0/T=j_0/j$. The boundary conditions for $g(\eta)$ are $g=0$ at $\eta=1$; $g_\eta=0$ at $\eta=0$. For $j(\eta)$ we have $j=i/U^2=$free stream value at $\eta=1$; at $\eta=0$ (i.e., on the plate) the boundary condition depends on the physical problem (i.e., heated or unheated plate). In particular, for the insulated plate $j_\eta=0$ at $\eta=0$. After $g(\eta)$ and $j(\eta)$ are known g is obtained from

$$y=(2\mu_0x/U\rho_0)\int_0^\eta(j/j_0)^\omega g^{-1}d\eta.$$

The skin friction coefficient c_f is given by

$$c_f=8^{\frac{1}{2}}g(0)\mu_0(\rho_\infty UL)^{-1}(U\rho_0L/\mu_0)^{\frac{1}{2}}.$$

The solution of the problem depends upon the values adopted for σ and ω. The form of the equation (*) is very convenient for discussing the influence of σ and ω.

The present paper gives a solution for the insulated plate with $\sigma=0.7$ and $\omega=0.8$. An extensive set of solutions for plates with heat transfer is given. The influence of the Mach number at given σ and ω is studied as well as the influence of ω for $\sigma=1$ and of σ for $\omega=1$. The influence of radiation on the thermal equilibrium is discussed briefly. *H. W. Liepmann* (Pasadena, Calif.).

Hantzsche, W., und Wendt, H. Die laminare Grenzschicht bei einem mit Überschallgeschwindigkeit angeströmten nichtangestellten Kreiskegel. Jahrbuch 1941 der Deutschen Luftfahrtforschung, I76–I77 (1941).

The pressure along the surface of a cone at zero angle of attack in supersonic flow is well known to be constant. Hence the pressure distribution for the cone is the same as for the flat plate. The boundary layer equations for the cone are:

$$\rho\left[U\frac{\partial U}{\partial r}+\frac{1}{r}V\frac{\partial U}{\partial\theta}\right]=\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\mu\frac{\partial U}{\partial\theta}\right),$$

$$\frac{\partial\rho U}{\partial r}+\frac{1}{r}\frac{\partial\rho V}{\partial\theta}+\frac{2\rho U}{r}=0,$$

$$\rho\left[U\frac{\partial i}{\partial r}+\frac{1}{r}V\frac{\partial i}{\partial\theta}\right]=\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\lambda}{c_p}\frac{\partial i}{\partial\theta}\right)+\frac{\mu}{r^2}\left(\frac{\partial U}{\partial\theta}\right)^2;$$

r, θ are polar coordinates about the apex of the cone, i is the enthalpy; U, V, ρ, c_p, λ, μ have their usual meaning. With $\chi=r(\theta-\theta_0)/r^{\frac{1}{2}}$, where θ_0 is the cone angle, the system of equations is reduced to a system of total differential equations for $U=U(\chi)$, $V=\overline{V(\chi)}/r^{\frac{1}{2}}$, $i=i(\chi)$:

$$\left[\frac{\rho U\chi}{2}+\rho\bar{V}\right]\frac{dU}{d\chi}=\frac{d}{d\chi}\left[\mu\frac{dU}{d\chi}\right],$$

$$-\frac{\chi}{2}\frac{d\rho U}{d\chi}+\frac{d\rho\bar{V}}{d\chi}+2\rho U=0,$$

$$\left[\frac{\rho U\chi}{2}+\rho\bar{V}\right]\frac{di}{d\chi}=\frac{d}{d\chi}\left[\frac{\lambda}{c_p}\frac{di}{d\chi}\right]+\mu\left[\frac{dU}{d\chi}\right]^2.$$

These equations differ from the equations for the flat plate only due to the additional term in the continuity equation. The authors show that the substitution $\bar{\chi}=3^{\frac{1}{2}}\chi$, $\bar{V}(x)=3^{\frac{1}{2}}[W(\bar{\chi})-\frac{1}{3}\bar{\chi}U]$ renders the equations for the cone and flat plate alike. Since the boundary conditions are similar it follows that the boundary layer solutions for the cone in supersonic flow can be obtained from the flat plate solutions. Specifically, if subscripts (c) and (p) refer respectively to cone and plate, $\delta_{(c)}=3^{-\frac{1}{2}}\delta_{(p)}$, $c_{f(c)}=\frac{2}{3}\cdot3^{\frac{1}{2}}c_{f(p)}$.
H. W. Liepmann (Pasadena, Calif.).

Prandtl, L., und Wieghardt, K. Über ein neues Formelsystem für die ausgebildete Turbulenz. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1945**, 6–19 (1945).

The older theories of turbulence have to treat various types of fully developed turbulence separately. For example, the ideas used in the usual momentum transfer theory cannot be applied to isotropic turbulence, where the mean flow has

no velocity gradient. The purpose of the present paper is to establish differential relations for the turbulent strength and thus obtain a formal system to include all types of fully developed turbulence.

The two fundamental equations developed are the energy equation

$$\frac{DE}{Dt}=-c\frac{E\sqrt{E}}{l}+kl\sqrt{E}\left(\frac{dU}{dy}\right)^2+\frac{\partial}{\partial y}\left(k_1l\sqrt{E}\frac{\partial E}{\partial y}\right),$$

and the familiar formula for Reynolds sheer,

$$\tau'=\epsilon dU/dy=k_e\sqrt{E}dU/dy;$$

where E is the kinetic energy of turbulent fluctuation per unit volume, D/Dt denotes substantial differentiation following the mean motion, l is the mixture length, c, k and k_1 are constants, U is the mean velocity in the direction of the x-axis, τ' is the Reynolds stress (divided by density), and ϵ is the exchange coefficient. It is shown how existing formulae can be derived from these as special cases. Wieghardt made an estimate of c from the decay of isotropic turbulence and obtained a value of 0.17–0.21. For flow through a channel, satisfactory agreement with experiments is obtained, especially for distribution of turbulence strength, with $c=0.18$, $k=0.56$, $k_1=0.38$. It is noted that $c=k^3$, in agreement with the general deductions of Prandtl in the first part of the paper. *C. C. Lin* (Cambridge, Mass.).

Obuhov, A. M. Turbulence in an atmosphere with inhomogeneous temperature. Akad. Nauk SSSR. Trudy Inst. Teoret. Geofiz. 1, 95–115 (1946). (Russian)

The author considers the problem of the variation of K, the kinematic coefficient of viscosity (coefficient of eddy diffusivity, turbulence coefficient), in a nonhomogeneous atmosphere. He assumes that K can be represented in the form $K=\varphi(Ri)K_0$, where φ is a dimensionless monotone decreasing "universal function" of the (dimensionless) Richardson number

$$Ri=\frac{g}{T}\frac{\partial\theta}{\partial z}\Big/\left(\frac{\partial v}{\partial z}\right)^2$$

and K_0 is the value of K in an atmosphere with dry-adiabatic lapse rate. These assumptions imply directly the boundary conditions $\varphi(0)=1$, $\varphi(Ri)=0$ for $Ri\geqq(Ri)_{cr}$, where $(Ri)_{cr}$ is the critical value of the Richardson number.

By subjecting K to known restrictions, e.g., energy balance, and by assuming that the shear stress and mean flux of heat in the turbulent layer are constant, it is found that $\varphi(Ri)=\{1-Ri/(Ri)_{cr}\}^{\frac{1}{2}}$ for $Ri\leqq(Ri)_{cr}$. Computations based on Sverdrup's value of $(Ri)_{cr}=1/11$ give K the correct order of magnitude and a reasonable distribution in layers up to 100 meters thickness for both stable and unstable temperature lapse rates. No quantitative check of φ against actual observed data has been made. *W. D. Duthie.*

Krasil'nikov, V. A. On the fluctuations of the amplitude of a sound in its propagation through a turbulent atmosphere. Doklady Akad. Nauk SSSR (N.S.) 58, 1353–1356 (1947). (Russian)

Primakoff, Henry, and Keller, Joseph B. Reflection and transmission of sound by thin curved shells. J. Acoust. Soc. Amer. 19, 820–832 (1947).

The authors discuss the reflection and transmission properties of thin curved shells by sound waves. First an inhomogeneous integral equation is derived for the sound field in an infinite medium containing a thin curved shell of different material. By appropriate approximations, the solution of the integral equation is reduced to the evaluation of a surface integral. This surface integral is similar to one obtained from the usual Kirchhoff diffraction theory. The integral is evaluated approximately and provides a basis for calculating the various physical parameters. *A. E. Heins.*

Vogel, Théodore. Étude, en deuxième approximation, de la transparence acoustique d'une plaque rectangulaire. Ann. Physique (12) 2, 502–516 (1947).

The author has recently studied the effect of sound waves on a rectangular elastic plate which is bordered by an infinite rigid wall [J. Phys. Radium (8) 7, 193–201 (1946)]. The excitation device was a plane wave whose propagation normal is perpendicular to the wall and the transmission properties were discussed. Now the author considers oblique, rather than normal, incidence and also discusses finite amplitudes of vibration. A method of normal coordinates is employed in this study. *A. E. Heins* (Pittsburgh, Pa.).

Elasticity, Plasticity

Moufang, R. Volumentreue Verzerrungen bei endlichen Formänderungen. Z. Angew. Math. Mech. 25/27, 209–214 (1947). (German. Russian summary)

Extending results obtained in an earlier paper [Ber. Math.-Tagung Tübingen 1946, pp. 109–110; these Rev. 9, 119], the author discusses the decomposition of the strain tensor into the strain deviation and a spherical tensor in the case of finite strains. The intensity of this spherical tensor is irrational in the invariants of the strain tensor. Approximating rational expressions are constructed, the simplest of which is the one familiar from the case of infinitesimal strains. *W. Prager* (Providence, R. I.).

Rozovskiĭ, M. I. Plane deformations in the presence of elastic after-effects and thermal stresses. Doklady Akad. Nauk SSSR (N.S.) 58, 999–1002 (1947). (Russian)

The author generalizes the Hooke-Volterra stress-strain relations in a long homogeneous isotropic elastic cylinder by including terms accounting for the thermal stresses. He is led to an integro-differential equation for the displacements, which is solved in terms of the resolvents of certain kernels. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Mandel, Jean. Sur les lignes de glissement et le calcul des déplacements dans la déformation plastique. C. R. Acad. Sci. Paris 225, 1272–1273 (1947).

In two previous notes [same C. R. 206, 317–318, 583–585 (1938)] the author discussed statically determinate stress distributions in a plastic material engaged in two-dimensional flow, the plastic behavior of the material being characterized by the envelope of the Mohr circles which represent limiting states of stress. In the present note this investigation is extended to the determination of the velocity distribution from appropriate boundary conditions. *W. Prager* (Providence, R. I.).

Arutyunyan, N. H. On the torsion of a quadrilateral bounded by elliptical arcs and radial lines. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 543–546 (1947). (Russian)

The note is concerned with the torsion problem of homogeneous anisotropic rectilinear rods, whose cross sections are quadrilaterals bounded by the radial lines $\rho=$constant and

elliptical arcs $\varphi=$constant. The curvilinear coordinates ρ and φ are connected with the Cartesian variables x, y by the formulas $x=a\rho^{\frac{1}{2}}\cos\varphi$, $y=b\rho^{\frac{1}{2}}\sin\varphi$, $a>b>0$. The determination of the stress function u reduces to the integration of $a_1u_{xx}+a_2u_{yy}+2=0$, subject to the restriction $u=0$ on the boundary, where a_1 and a_2 are the elastic constants of the medium, supposed to be orthotropic. The problem is solved approximately by minimizing a suitable energy integral. The paper contains several numerical examples permitting comparisons with known results. *I. S. Sokolnikoff.*

Ruhadze, A. K. On the deformation of naturally twisted bars. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 533–542 (1947). (Russian)

The problem of equilibrium of long naturally twisted rods of an arbitrary cross section was considered by P. M. Riz [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 449–476; these Rev. **1**, 287] and by Lourie and Janelidze [C. R. (Doklady) Acad. Sci. URSS (N. S.) **24**, 24–27, 227–228 (1939); **27**, 436–439 (1940); these Rev. **2**, 176]. Riz considered the problems of extension, torsion and bending by couples by a method of perturbation of a small parameter characterizing the uniform twist. Lourie and Janelidze used the same method to solve the problem of bending by a transverse force and found that their results appeared in a form inconvenient for computations. The author [same journal **6**, 123–138 (1942); these Rev. **4**, 180] proposed a method of solution of these problems using the theory of functions of a complex variable. The case of uniform twist, determined along the axis of the rod by the function $\theta=kz$ with parameter k so small that one can neglect its second and higher powers, was considered. This paper contains an explicit solution of the problem of bending by a transverse force of naturally twisted rods, of elliptical cross section, by the method developed in the author's earlier article. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Federhofer, Karl. Die Grundgleichungen für elastische Platten veränderlicher Dicke und grosser Ausbiegung. Z. Angew. Math. Mech. **25/27**, 17–21 (1947).

The author generalizes the known system of two nonlinear differential equations for finite deflections of thin plates so as to include the effect of gradually varying thickness of the plate. The results are given for Cartesian as well as for polar coordinates. *E. Reissner.*

Flügge, W. Zur Membrantheorie der Drehschalen negativer Krümmung. Z. Angew. Math. Mech. **25/27**, 65–70 (1947). (German. Russian summary)

It is pointed out the character of the differential equations of equilibrium of the linear membrane theory of shells of revolution is importantly affected by the sign of the product of the two principal radii of curvature of the shell. When this product is positive, as for the spherical shell, the equations are of elliptic type. When the product is negative, as for a shell having the form of a hyperboloid of revolution, the equations are of hyperbolic type. The latter case is discussed in some detail so as to bring out some of the consequences of the fact that the differential equations of the problem have real characteristics. *E. Reissner.*

Muštari, H. M. On the domain of applicability of the Kirchhoff-Love theory of shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 517–520 (1947). (Russian)

The author points out that some writers in the theory of shells introduce refinements which are superfluous when the Kirchhoff-Love hypothesis is used. *I. S. Sokolnikoff.*

Muštari, H. M. On the domain of applicability of the linear theory of elastic shells. Doklady Akad. Nauk SSSR (N.S.) **58**, 997–998 (1947). (Russian)

The author linearizes the equations of the shell theory developed by J. L. Synge and W. Z. Chien [von Kármán Anniversary Volume, pp. 103–120, Berkeley, Calif., 1941; these Rev. **3**, 30] and makes remarks concerning their validity. [Cf. the preceding review.] *I. S. Sokolnikoff.*

Kunce, I. P. The stability of a cylindrical shell beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 561–562 (1947). (Russian)

The paper is concerned with the rotationally symmetric buckling of an infinitely long shell under axial compression with or without simultaneous exterior pressure. The shell is assumed to consist of a perfectly plastic material which obeys Hencky's theory of deformation [Z. Angew. Math. Mech. **4**, 323–334 (1924)] and the initial axial stress is supposed to equal the yield stress. *W. Prager.*

Bijlaard, P. P. On the plastic stability of thin plates and shells. Nederl. Akad. Wetensch., Proc. **50**, 765–775 (1947).

In earlier papers [same Proc. **41**, 468–486, 731–743 (1938)] the author developed a stress-strain law relating the increments of strain in the plastic range to the stresses, strains and increments of stress, and applied this law to the plastic buckling of plates. In the first part of the present article, the theoretical results of the earlier papers are compared with the results of recent Swiss experiments. The agreement between theory and experiment is found to be reasonably good provided that it is assumed that no unloading occurs when the plate buckles. The author's stress-strain law is then applied to the buckling of a cylindrical shell under axial compression. *W. Prager.*

Galin, L. A. The pressure of a punch with a plane base, in the form of an infinite wedge, on an elastic half space. Doklady Akad. Nauk SSSR (N.S.) **58**, 205–208 (1947). (Russian)

The distribution of stresses in a semi-infinite isotropic elastic half-space $z\geqq 0$ produced by the pressure of a rigid punch bounded by a contour C reduces to the search for a harmonic function $\varphi(x, y, z)$, regular at infinity, and satisfying the conditions $\varphi=c$ in the region interior to C and $\partial\varphi/\partial z=0$ in the region exterior to C. The constant c denotes the displacement of the punch in the direction of the z-axis. The author solves this boundary value problem for a wedge-shaped region formed by a pair of straight lines intersecting at an angle α. *I. S. Sokolnikoff* (Los Angeles, Calif.).

McLachlan, N. W. Vibrational problems in elliptical coordinates. Quart. Appl. Math. **5**, 289–297 (1947).

The author starts from the familiar two-dimensional wave equation in rectangular coordinates upon transformation of which to elliptical coordinates and separation of the variables, the familiar Mathieu and associated Mathieu differential equations are obtained. Using these equations the author sets out to derive formal solutions pertaining to the vibrational modes of (a) a uniform homogeneous loss-free stretched membrane in the form of an elliptical ring; (b) water in a lake of uniform depth whose plan view is an

elliptical ring; (c) a uniform homogeneous loss-free elastic elliptical plate; (d) a uniform homogeneous loss-free elastic elliptical ring plate. In the case of the elliptical ring membrane the formal solution is derived from the conditions of the problem and the vibrational modes are discussed. A similar solution is given in the other two cases.

M. J. O. Strutt (Eindhoven).

White, Walter T. An integral-equation approach to problems of vibrating beams. I. J. Franklin Inst. **245**, 25–36 (1948).

This is a detailed exposition on the reduction of the (one-dimensional) vibrating beam problem to a homogeneous Fredholm integral equation for the deflection function and its solution. The well-known iteration technique is discussed and an example is given. *G. F. Carrier.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

***Bouwers, A. Achievements in Optics.** Monographs on the Progress of Research in Holland During the War, vol. 1. Elsevier Publishing Company, Inc., New York and Amsterdam, 1946. viii+136 pp. $2.50.

The material is arranged in four chapters: New optical systems, New optical instruments, Geometrical optics, Physical optics. In the first chapter a comprehensive theory is given of optical systems which contain a spherical mirror as element. The advantage of combining a strong spherical mirror with weak refracting elements has been well known since the development of the Schmidt camera. The author shows that it is possible to replace the aspheric corrector plate of the Schmidt system by one or more spherical lenses. Especially good results and simple design methods are found if the refracting elements are concentric to a common center which is also the center of the mirror. The performance of such a system is investigated theoretically and illustrated by numerical examples.

The results of the first chapter lead to several new designs of optical instruments. These are described in chapter 2. Two mirror microscopes could be developed which have extremely simple objectives and are adequate for magnifications of 200 and 300 and for a numerical aperture of 0.2 or 0.3. Other technical results are represented by several new telescopes and a high aperture photographic camera. All these designs use the optical mirror systems derived in chapter 1.

Chapter 3 gives a summary of results in geometrical optics obtained by several authors and published elsewhere. A short section deals with graphical methods of ray tracing through spherical surfaces [L. E. W. Albada, Optische stelsels, 1944]. On the basis of former investigations by T. Smith certain advances in the theory of third and fifth order aberrations have been made by J. Korringa [Thesis, Delft, 1942] and by W. G. Stephan [unpublished]. A review of these results is given. Another section of the chapter deals with an approach by B. R. A. Nijboer [Thesis, Groningen, 1942] to the theory of aberrations. The intersection of rays with the image plane is developed into series of trigonometric functions with regard to the polar angle φ of the ray. This method gives a natural classification of aberrations and also a means of evaluating phase integrals [chapter 4].

The last chapter contains, in addition to a second extract from Nijboer's thesis, a reprint of the paper by F. Zernike in which the method of phase contrast was introduced [Physica **9**, 686–693, 974–986 (1942)]. Meanwhile, this method has become a significant tool in microscopy. Zernike gives first a clear exposition of the principles of other methods which have been used to make transparent objects visible [Schlieren method, dark ground illumination, etc.]. Then the method of phase contrast is derived and its principles are explained. The method consists in inserting an absorbing or retarding phase plate upon the geometrical image of the light source. This has the effect that a purely retarding object is imaged as an absorbing object and thus becomes visible. The efficiency of the method is illustrated by several microphotographs. *R. K. Luneburg.*

Lansraux, Guy. Calcul des figures de diffraction des pupilles de révolution. Rev. Optique **26**, 278–294 (1947).

In an earlier paper [same vol., 24–45 (1947); these Rev. **8**, 422] the author obtained a series expression for the complex displacement $G(r)$ at a point P in the image-plane by expanding the wave-function $F(\rho)$ in the exit-pupil $0 \leqq \rho \leqq R$ as a Taylor series

$$F(\rho) = \sum_{p=1}^{\infty} a_p (1-\rho^2/R^2)^{p-1} \tag{1}$$

in powers of $1-\rho^2/R^2$. His result was

$$G(r)/(\pi R^2) = \Gamma(W) = \sum_{p=1}^{\infty} p^{-1} a_p L_p(W), \tag{2}$$

where (3) $L_p(W) = 2^p p! W^{-p} J_p(W)$, $W = 2\pi Rr$, R is the radius of the exit-pupil in wavelengths, r the sine of the off-axis angle of P subtended at the centre of the exit-pupil.

The present paper described a technique for the approximate computation of $G(r)$ in a practical case. Here $F(\rho)$ is replaced by the first p terms of its expansion (1) and the values of the resulting polynomial $f(\rho)$ are calculated for p special values of ρ. The values of the corresponding approximation to $\Gamma(W)$, namely $\gamma(W) = \sum_{r=1}^{p} f(\rho_r) \mathfrak{L}_r^p(W)$, are then given by an equation $\gamma(W) = \sum_{r=1}^{p} f(\rho_r) \mathfrak{L}_r^p(W)$, in which the functions $\mathfrak{L}_r^p(W)$ depend on the functions (3) and on the choice of the p special values $\rho_1, \cdots, \rho_p$ of ρ, but not on f. Some discussion is made of the accuracy of the approximation in $\Gamma(W)$ corresponding to given accuracy of approximation to $F(\rho)$, and evidence is brought that the new method yields a given accuracy with less labour than the more obvious procedure of evaluating the integral

$$\int_0^1 F(\rho) J_0(W\rho/R) d(\rho^2/R^2) = \Gamma(W)$$

by Cotes's formula. *E. H. Linfoot* (Bristol).

Pidduck, F. B. Diffraction of light. Philos. Mag. (7) **38**, 439–441 (1947).

The object of this note is to correct an error in a recent paper [same Mag. (7) **37**, 280–287 (1946); these Rev. **8**, 363] and to describe some experiments on Talbot's bands.

A. E. Heins (Pittsburgh, Pa.).

Arzeliès, Henri. Étude de l'onde obtenue par réflexion vitreuse totale dans les milieux à susceptibilité magnétique non nulle. Ann. Physique (12) **2**, 517–535 (1947).

This paper forms a sequel to a previous account [same Ann. (12) **1**, 5–69 (1946); these Rev. **9**, 125] of total internal reflection. The author examines the case of plane electromagnetic waves meeting a plane boundary surface between two media of nonzero magnetic susceptibilities. He points out, in particular, that it is not sufficient, in order to obtain correct results, to replace n by $(\kappa\mu)^{\frac{1}{2}}$ in the classical Fresnel formulae. *E. H. Linfoot* (Bristol).

Arzeliès, Henri. Sur le calcul de l'énergie électromagnétique dissipée dans un milieu absorbant sélectif. Ann. Physique (12) **2**, 536–554 (1947).

In an earlier paper [same Ann. (12) **2**, 133–194 (1947); these Rev. **9**, 125] the author proposed to replace the classical theory of reflection, based on the use of an imaginary refractive index, by one based on the movements of ions in the media. The present paper applies the proposed theory to study the energy-dissipation of plane electromagnetic waves in certain selectively absorbing media, which include perfect conductors as a special case. The results are applied to a brief discussion of energy-dissipation in plane laminae and in prisms. *E. H. Linfoot* (Bristol).

Copson, E. T. On the problem of the electrified disc. Proc. Edinburgh Math. Soc. (2) **8**, 14–19 (1947).

The problem of a perfectly conducting thin circular disc maintained at potential V_0 in an external electrostatic field of potential Φ is a boundary value problem which may be formulated as a pair of dual integral equations. That is, if $J_n(\rho)$ is customary notation for the Bessel function, the above problem may be formulated as the following pair of integral equations:

$$\text{(i)} \qquad \int_0^\infty \psi(t)J_n(\rho t)\,dt = f(\rho), \qquad 0<\rho<a,$$

$$\text{(ii)} \qquad \int_0^\infty t\psi(t)J_n(\rho t)\,dt = 0, \qquad \rho>a,$$

where a is the radius of the disc, $\psi(t)$ is a function from which the surface charge density may be computed and $f(\rho)$ is known. Several methods of solution have been given for the equations (i) and (ii). The author shows that a reformulation of the problem enables one to obtain a single integral equation which can be solved by two applications of the known solution of Abel's integral equation. *A. E. Heins* (Pittsburgh, Pa.).

Auner, Michael. Ein Beitrag zur Theorie ebener Magnetfeldröhren. Akad. Wiss. Wien, S.-B. IIa. **152**, 143–172 (1943).

The paper deals with a plane magnetron tube. The author has divided his work into three parts. In the first part he investigates the variation of the anode current J with the change of the applied magnetic field H, which is perpendicular to the electrode surfaces. Here it is assumed that the electrons emitted from the cathode have either zero or constant velocities in the direction y, perpendicular to H. The effect of space charge is included. With the assumption that both the velocity, $\dot{y}^2$, and $d\dot{y}^2/dy$ vanish only at the anode, the author integrates the potential equation and obtains a functional relation between the current density J and the applied field H. The curve is represented by a transcendental equation in $L=J/J_e$ and $N=H/H_c$, where J_e is the saturation current at the cathode and H_c the magnetic field where $\dot{y}^2=0$ and $d\dot{y}^2/dy=0$. This position is called the virtual cathode of the magnetron. For the case where there exist a virtual cathode between the cathode and the anode the integration of the potential equation is split into two parts, one region of integration being from cathode to virtual cathode and the other from virtual cathode to anode. From the two solutions the potential f and the distance d are eliminated and a transcendental equation between L and N and similar to that for the previous case is obtained. The L-N curve decreases monotonically as N increases from zero to one. At $N=1$ the curve suddenly drops to zero with a slope of $-\infty$. If one assumes constant initial velocities for the emitted electrons, the shape of the L-N curve does not vary much from that of the preceding case: however, here L vanishes for a value of N slightly greater than one; and at $N=1$ the slope is still negative but has a finite value.

The second part contains an analysis of the case where a Maxwellian distribution of initial velocities is assumed and the space charge is neglected. The L-N curve remains practically constant till N approaches unity, then it drops steeply to zero for N slightly larger than one. The slope of the curve at $N=1$ is finite. The slope of the L-N curve for $L=0$ is not infinite as in the previous case, but has a finite negative value. In the third part the author includes the space charge. For this case the electrons can have zero velocities $\dot{y}^2=0$, $d\dot{y}^2/dy=0$ (minimum) at some point between the cathode and the anode, as well as at the anode itself. Since one does not know the region of integration, the integrodifferential equation for the potential cannot be solved and therefore the elimination of f and d cannot be effected in order to obtain a functional relation between L and N. However, for zero magnetic field the virtual cathode is near the cathode. When the field H is increased the virtual cathode approaches the cathode and for strong fields it is practically at the cathode. It is supposed that $f_{\min}$ and $d_{\min}$ are known ($f_{\min}<0$). Thus two positions are assumed where $\dot{y}^2$ and $d\dot{y}^2/dy$ vanish, i.e., one at $d_{\min}$ and the other at the anode. This being assumed, the region R is determined and the integration of the potential equation is effected. However, $f_{\min}$ and $d_{\min}$ are not known and one has to determine them from experimental data in order to evaluate L and therefore the current density J. Finally several appendices are added where the mathematical calculations are given. *N. Chako* (Auburn, Ala.).

Rytov, S. M. Certain theorems on the group velocity of electromagnetic waves. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 930–936 (1947). (Russian)

This paper deals with the propagation of a quasi-monochromatic, quasi-plane wave in a linear, quasi-homogeneous and quasi-static, nonabsorbing, dispersive medium endowed with an arbitrary electric and magnetic anisotropy and rotatory power. The theorems proved consist of the following set of 24 relations between the complex amplitudes F_{ik}, H^{lm} of the field components, the components of the propagation four-vector $\kappa_m=(\mathbf{k}, -\omega(\mathbf{k})/c)$, and the complex permittivity-permeability tensor defined by $H^{lm}=\epsilon^{lmik}F_{ik}$:

$$\frac{\bar{F}_{kl}H^{lm}+F_{kl}\bar{H}^{lm}}{2}\frac{\partial\kappa_m}{\partial\kappa_a}-\frac{\kappa_k}{4}\frac{\partial\epsilon^{lmpq}}{\partial\kappa_4}\bar{F}_{lm}F_{pq}\frac{\partial\kappa_4}{\partial\kappa_a}=0,$$

$$\frac{\bar{F}_{kl}H^{lm}-F_{kl}\bar{H}^{lm}}{2}\frac{\partial\kappa_m}{\partial\kappa_a}+\frac{\kappa_k}{4}\left(H^{lm}\frac{\partial\bar{F}_{lm}}{\partial\kappa_a}-\bar{H}^{lm}\frac{\partial F_{lm}}{\partial\kappa_a}\right)=0,$$

with $k=1, 2, 3, 4$ and $\alpha=1, 2, 3$. The first 12 relations are given the three-dimensional interpretation:

$$\theta_{\alpha\beta}=-g_\alpha u_\beta, \quad S_\alpha=wu_\alpha, \qquad \alpha, \beta=1, 2, 3,$$

where the three-vector $u_\alpha=\partial\omega(\mathbf{k})/\partial k_\alpha$ is the group velocity and $\theta_{\alpha\beta}$, S_α, g_α, w are respectively the time averages over the fundamental period of the wave group of the Maxwell stress tensor, the Poynting flux vector, the momentum density and the energy density. The author is unable to find a similar simple three-dimensional interpretation of the second set of 12 relations involving the derivatives of the field amplitudes F_{lm} with respect to κ_α. *G. M. Volkoff.*

Kacenelenbaum, B. Z. On the propagation of electromagnetic waves along infinite dielectric cylinders at low frequencies. Doklady Akad. Nauk SSSR (N.S.) **58**, 1317–1320 (1947). (Russian)

Rigal, Roger, et Voge, Jean. Sur la propagation d'ondes planes dans un guide métallique droit de section quelconque. C. R. Acad. Sci. Paris **226**, 326–328 (1948).

Graffi, Dario. Sul teorema di reciprocità per le correnti elettriche variabili. Ann. Mat. Pura Appl. (4) **25**, 267–276 (1946).

The author derives a reciprocity theorem for linear electric circuits valid for general variable currents with arbitrary initial values. The Laplace transform is used.

I. Opatowski (Ann Arbor, Mich.).

Quantum Mechanics

***Dirac, P. A. M. The Principles of Quantum Mechanics.** 3d ed. Oxford, at the Clarendon Press, 1947. xii+311 pp. $9.00.

Dirac's book is one of the classics in the now immense literature of quantum theory. By means of a symbolic algebra, the physical bases of quantum mechanics are formalized and their consequences carried out with precision but without the digressive rigor usually employed by mathematicians.

In the present edition Dirac has chosen to concentrate mainly upon exposition, rather than expanding the book to include, except as noted below, new fundamental material which has made its appearance since the publication of the second edition in 1935. Perhaps the most radical change has been the introduction of a new notation, the "bra" and "ket," whose chief effect is to render the relation between states and wave functions more transparent. Many of the proofs become shorter and clearer. In addition to the notation, the whole logical chain has been reorganized and strengthened; theorems have been reformulated, proofs sharpened or replaced. This reviewer enjoyed particularly the discussion of the existence of eigenvalues and the corresponding completeness theorems, the use of dyadics in vector space to represent operators, the discussion of displacement operators, etc. A considerable improvement has been made in the treatment of second quantization, particularly the transition from configuration space to number space and the relation of the latter to assemblies of harmonic oscillators.

The physical content of the third edition does not differ significantly from that of the second. This is of course a reflection of the fact that the foundations of quantum mechanics have become more solidly entrenched in the last thirteen years. Actually most of the recent attempts to reformulate quantum mechanics have not altered the fundamental physical principles in any way, but rather have attempted to remove the Hamiltonian from the central role it now plays in the present formulation.

The frontiers of quantum theory, particularly in its application to fields, are briefly discussed in the chapter entitled "Quantum electrodynamics." The chief additions here include a discussion of the classical motion of a charged particle and the Wentzel λ-limiting process. The many-time treatment of the second edition has been somewhat reduced. Recent developments stimulated by the experiments of Lamb and Retherford have made the admittedly provisional Wentzel treatment somewhat obsolete, less than a year after the publication of this book. However, the explorations of the relativistic formulation of quantum mechanics are playing a useful part in the most recent work in this field. This perhaps gives us a clue to Dirac's reluctance to include some of the developments in positron theory, theories of particles with spins other than $\frac{1}{2}\hbar$ and the relation between spin and statistics, for these are of a provisional nature also. *H. Feshbach* (Cambridge, Mass.).

d'Espagnat, Bernard. Un procédé simple pour l'étude de certains problèmes d'évolution, avec application au cas d'une particule mobile en l'absence de champ. C. R. Acad. Sci. Paris **225**, 1058–1059 (1947).

The author extends the notion of the characteristic function $\Phi(u, v)$ of two operators A and B, previously defined when A and B are permutable, to the case where $AB-BA$ is a number. The definition is formally the same as before, namely $\Phi(u, v)=(\psi, \exp[2\pi i(uA+vB)]\psi)$, where ψ is the wave function of a state. If ψ_t represents the value of ψ at time t, then without the work of calculating ψ_t from its given initial value ψ_0 it is shown that physical conclusions derivable from a knowledge of ψ_t can be obtained easily and directly from the characteristic function. This is illustrated for the cases of a free particle and a wave packet.

O. Frink (State College, Pa.).

d'Espagnat, Bernard. Application à l'étude de l'oscillateur harmonique de la fonction caractéristique quantique. C. R. Acad. Sci. Paris **226**, 316–318 (1948).

The method of characteristic functions [see the preceding review] is applied to the quantum theory of the linear harmonic oscillator. The author concludes that, except in the special case considered by Schrödinger, the probability wave packet associated with the particle has, in addition to its oscillation with the classical frequency, an internal pulsation of twice this frequency. *O. Frink.*

Viard, Jeannine. Sur deux méthodes simples d'intégration en mécanique ondulatoire des systemes: Intégrales premières; petits mouvements autour d'une position d'équilibre stable. Cahiers de Physique, no. **28**, 68–96 (1945).

This is an essentially expository article, dealing with certain analogies between classical mechanics and wave mechanics. In the first part the concepts of first integrals in the two theories are defined and compared, and the applications of first integrals to the solution of dynamical problems are discussed. The second part is devoted to a discussion of the wave mechanical analogue of the classical theory of small vibrations about a configuration of equi-

librium. In the third part the results of the first two parts are used in a short treatment of the n-body problem in wave mechanics. Substantially all of the contents of the paper are to be found, at least in implicit form, in the standard treatises. *L. A. MacColl* (New York, N. Y.).

March, Arthur. Quantentheorie der Wellenfelder und kleinste Länge. I. Acta Physica Austriaca 1, 19–41 (1947).

A minimum length l_0 is postulated such that it is physically meaningless to regard as spatially separate two particles at rest unless their distance apart exceeds l_0. For a common velocity v of the particles the minimum length becomes $l_0(1-v^2/c^2)^{\frac{1}{2}}$. Particles are interpreted as having, not material extension, but a "field of influence" of dimensions l_0. Similarly a smallest measurable time $t_0=l_0/c$ is postulated. The limitation $|(\Delta p)^2-(\Delta E/c)^2|<(h/l_0)^2$ proposed by Heisenberg [Z. Physik 110, 251–266 (1938)] on the increments Δp of momentum, ΔE of energy for a particle in an elementary process, is always valid for experiments which, without this limitation, would measure arbitrarily small lengths or times, but not necessarily valid for other processes. Correspondingly a particle emitted by a particle at rest has an upper limit to its mass: identification with the meson mass gives $l_0\sim 5.0\times 10^{-13}$ cm. Charge density is supposed distributed over a sphere (or spheroid for a moving particle), a statement which, in view of the metric, is not to be taken literally: however, self-energy calculations use ordinary integration over this distribution and give finite results. Reasons for the nonappearance of l_0 in experimental electron-physics, the scattering of light, the Compton effect and cavity radiation are discussed. *C. Strachan.*

March, Arthur. Quantentheorie der Wellenfelder und kleinste Länge. II. Acta Physica Austriaca 1, 137–154 (1947).

The relativistic invariance of the theory of the preceding review is discussed with respect to (1) the term in the Hamiltonian giving interaction of the particle with radiation and (2) the mutual Coulomb energy of two point charges. For (1) the value of a field quantity at a point charge is replaced by an expression involving a space-average over a volume defined in the rest-system of the charge. In quantum-mechanical transition theory two space-averages are used for the rest-systems of the particle in its initial and final states. The solution of (2) contradicts the geometry of special relativity without giving a consistent alternative. For electrons, small mass and consequent large increments of velocity with the resulting large Lorentz contractions reduce the consequences of the theory to minor significance. For nucleons this is not so. A nucleon at rest cannot emit or absorb mesons of momentum greater than h/l_0. This alters the interaction between nucleons through the meson field, the range of nuclear forces being determined by l_0 and not by the parameter of the meson field. Nuclear scattering of high energy mesons and the generation of a meson by interaction of a light-quantum with a nucleon are considered. *C. Strachan* (Aberdeen).

Blatt, John M. On the Heitler theory of radiation damping. Physical Rev. (2) 72, 466–477 (1947).

The integral equation characteristic of the theory of radiation damping is derived classically for a special group of problems and it is pointed out as an essential feature of the method that the part of the scattered wave which reacts on the scatterer is approximated by half the difference between a retarded and an advanced potential. This carries one's thoughts to the λ-limiting process but the author lays stress on a fundamental distinction between the two theories: in using the λ-process one is restricted to perturbation methods whereas the theory of radiation damping leads to a well-defined problem which is not linked to perturbation procedures.

The author goes on to compare the exact solution, the Born approximation and the Heitler solution of some simply constructed problems. One is the interaction of an oscillator with a scalar meson field, where the Heitler method turns out to preserve the shape of the resonance curve [scattering cross section vs. energy of incident mesons] whereas it fails to reproduce the shift in resonance frequency caused by the additional mass of the oscillator due to its coupling with the meson field. In the case of an oscillator interacting with a vector meson field the strong radiation damping appears, although no spin interaction is implied; the strong damping is shown to derive from the condition $\sum_{i=1}^4 \partial U_i/\partial x_i=0$ imposed on the field vector U_i [cf. W. Pauli, Rev. Modern Physics 13, 203–232 (1941), p. 213].

It is known that the theory discussed here leads to difficulties in the region of long wave-lengths [infra-red catastrophe reappears; cf. Bethe and Oppenheimer, Physical Rev. (2) 70, 451–458 (1946)]. An important conclusion of the present paper is that the Heitler method cannot even be used for short wave-lengths, if the concept of finite source is adopted. *L. Hulthén* (Lund).

Green, Alex E. S. On infinities in generalized meson-field theory. Physical Rev. (2) 73, 26–29 (1948).

The generalized electrodynamics developed by B. Podolsky [same Rev. (2) 62, 68–71 (1942); these Rev. 4, 31] and others is extended to the meson field theory, by considering a Lagrangian which depends explicitly upon the field coordinates, namely $L=-\frac{1}{2}\{\mu^2\varphi_\alpha^2+\varphi_{\alpha,\beta}^2+a^2(\Box\varphi_\alpha)^2\}$. It is shown that this corresponds to a field associated with particles of two different rest masses $m_0=\hbar\mu_0/c$, $m_1=\hbar\mu_1/c$, where

$$\mu_0=\{(1+2a\mu)^{\frac{1}{2}}-(1-2a\mu)^{\frac{1}{2}}\}/2a,$$
$$\mu_1=\{(1+2a\mu)^{\frac{1}{2}}+(1-2a\mu)^{\frac{1}{2}}\}/2a.$$

The interaction energy is calculated by a scheme given by B. Podolsky and C. Kikuchi [same Rev. (2) 67, 184–192 (1945); these Rev. 6, 283] and the author [same Rev. (2) 72, 628–631 (1947); these Rev. 9, 69]. The resulting expression is similar to the interaction function used in the "mixed" theories of meson forces. It is also shown that the self-energy is finite, in contrast with the results of most relativistic theories. [Reviewer's note. A field similar to the one suggested here was considered by D. Iwanenko and A. Sokolow, Acad. Sci. USSR. J. Phys. 8, 54–55, 358–360 (1944).] *C. Kikuchi* (East Lansing, Mich.).

Giao, Antonio. Sur la relation entre le moment magnétique et le moment de rotation des masses sphériques. C. R. Acad. Sci. Paris 225, 924–926 (1947).

Further consideration is given to the deduction of Blackett's relation between the magnetic moment and the moment of momentum of a rotating star from the author's cosmological theories [Portugaliae Phys. 2, 1–98 (1946); Portugaliae Math. 5, 145–193 (1946); same C. R. 224, 1813–1815 (1947); these Rev. 8, 121, 555; 9, 107]. *C. Strachan* (Aberdeen).